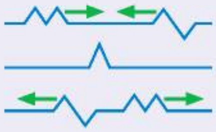
17 Superposition

This swirl of colors is due to a very thin layer of oil. Oil is clear. The colors arise from the interference of reflected light waves.

IN THIS CHAPTER, you will understand and use the ideas of superposition.

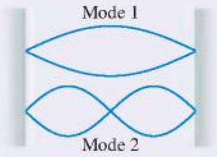
What is superposition?



Waves can pass through each other. When they do, their displacements add together at each point. This is called the principle of superposition. It is a property of waves but not of particles.

« LOOKING BACK Sections 16.1–16.4 Properties of traveling waves

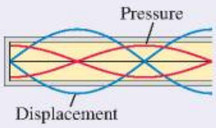
What is a standing wave?



A standing wave is created when two waves travel in opposite directions between two boundaries.

- Standing waves have well-defined patterns called modes.
- Some points on the wave, called nodes, do not oscillate at all.

How are standing waves related to music?



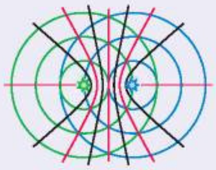
The notes played by musical instruments are standing waves.

- Guitars have string standing waves.
- Flutes have pressure standing waves.

Changing the length of a standing wave changes its frequency and the note played.

« LOOKING BACK Section 16.5 Sound waves

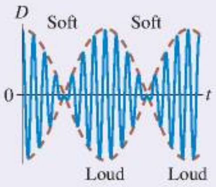
What is interference?



When two sources emit waves with the same wavelength, the overlapped waves create an interference pattern.

- Constructive interference (red) occurs where waves add to produce a wave with a larger amplitude.
- Destructive interference (black) occurs where waves cancel.

What are beats?



The superposition of two waves with slightly different frequencies produces a loud-soft-loud-soft modulation of the intensity called beats. Beats have important applications in music, ultrasonics, and telecommunications.

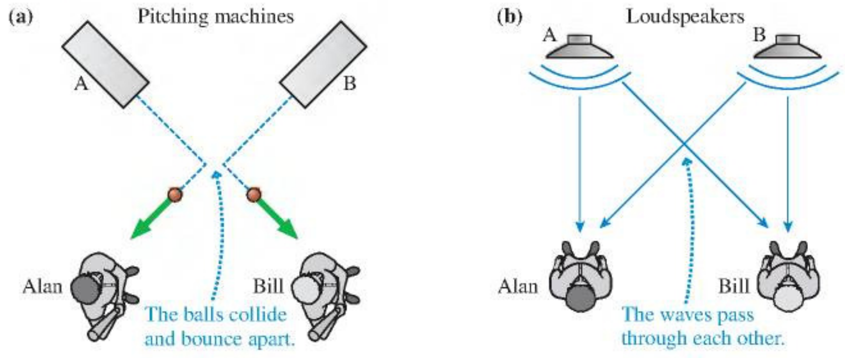
Why is superposition important?

Superposition and standing waves occur often in the world around us, especially when there are reflections. Musical instruments, microwave systems, and lasers all depend on standing waves. Standing waves are also important for large structures such as buildings and bridges. Superposition of light waves causes interference, which is used in electro-optic devices and precision measuring techniques.

17.1 The Principle of Superposition

FIGURE 17.1a shows two baseball players, Alan and Bill, at batting practice. Unfortunately, someone has turned the pitching machines so that pitching machine A throws baseballs toward Bill while machine B throws toward Alan. If two baseballs are launched at the same time, and with the same speed, they collide at the crossing point. Two particles cannot occupy the same point of space at the same time.

FIGURE 17.1 Unlike particles, two waves can pass directly through each other.

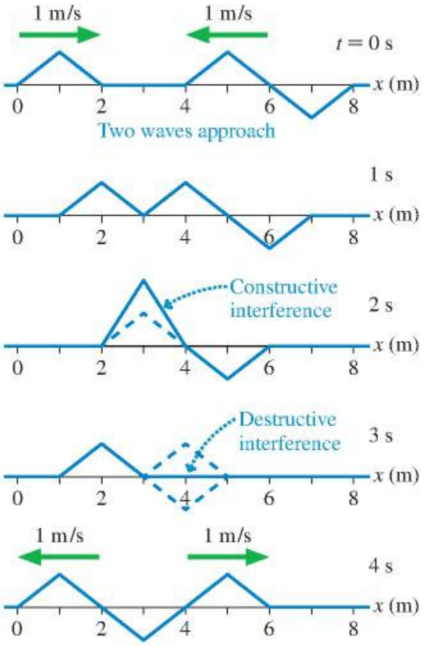


But waves, unlike particles, can pass directly through each other. In FIGURE 17.1b Alan and Bill are listening to the stereo system in the locker room after practice. Because both hear the music quite well, the sound wave that travels from loudspeaker A toward Bill must pass through the wave traveling from loudspeaker B toward Alan.

What happens to the medium at a point where two waves are present simultaneously? If wave 1 displaces a particle in the medium by D_1 and wave 2 *simultaneously* displaces it by D_2, the net displacement of the particle is simply $D_1 + D_2$. This is a very important idea because it tells us how to combine waves. It is known as the *principle of superposition*.

Principle of superposition When two or more waves are *simultaneously* present at a single point in space, the displacement of the medium at that point is the sum of the displacements due to each individual wave.

FIGURE 17.2 The superposition of two waves as they pass through each other.



Mathematically, the net displacement of a particle in the medium is

$$D_{\text{net}} = D_1 + D_2 + \cdots = \sum_i D_i \tag{17.1}$$

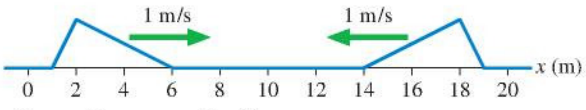
where D_i is the displacement that would be caused by wave i alone. We will make the simplifying assumption that the displacements of the individual waves are along the same line so that we can add displacements as scalars rather than vectors.

To use the principle of superposition you must know the displacement caused by each wave if traveling alone. Then you go through the medium *point by point* and add the displacements due to each wave *at that point* to find the net displacement at that point.

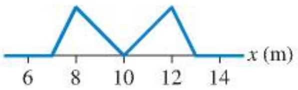
To illustrate, FIGURE 17.2 shows snapshot graphs taken 1 s apart of two waves traveling at the same speed (1 m/s) in opposite directions. The principle of superposition comes into play wherever the waves overlap. The solid line is the sum *at each point* of the two displacements at that point. This is the displacement that you would actually observe as the two waves pass through each other.

Notice how two overlapping positive displacements add to give a displacement twice that of the individual waves. This is called *constructive interference*. Similarly, *destructive interference* is occurring at the points where positive and negative displacements add to give a superposition with zero displacement. We will defer the main discussion until later in this chapter, but you can already see that *interference is a consequence of superposition*.

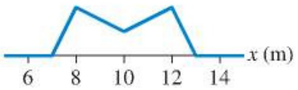
STOP TO THINK 17.1 Two pulses on a string approach each other at speeds of 1 m/s. What is the shape of the string at $t = 6$ s?



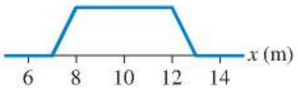
Approaching waves at $t = 0$ s



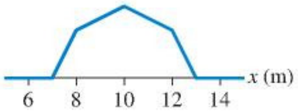
(a)



(b)



(c)



(d)

17.2 Standing Waves

FIGURE 17.3 is a time-lapse photograph of a *standing wave* on a vibrating string. It's not obvious from the photograph, but this is actually a superposition of two waves. To understand this, consider two sinusoidal waves **with the same frequency, wavelength, and amplitude** traveling in opposite directions. For example, **FIGURE 17.4a** shows two waves on a string, and **FIGURE 17.4b** shows nine snapshot graphs, at intervals of $\frac{1}{8}T$. The dots identify two of the crests to help you visualize the wave movement.

At *each point,* the net displacement—the superposition—is found by adding the red displacement and the green displacement. **FIGURE 17.4c** shows the result. It is the wave you would actually observe. The blue dot shows that the blue wave is moving neither right nor left. The wave of Figure 17.4c is called a **standing wave** because the crests and troughs "stand in place" as the wave oscillates.

FIGURE 17.3 A vibrating string is an example of a standing wave.

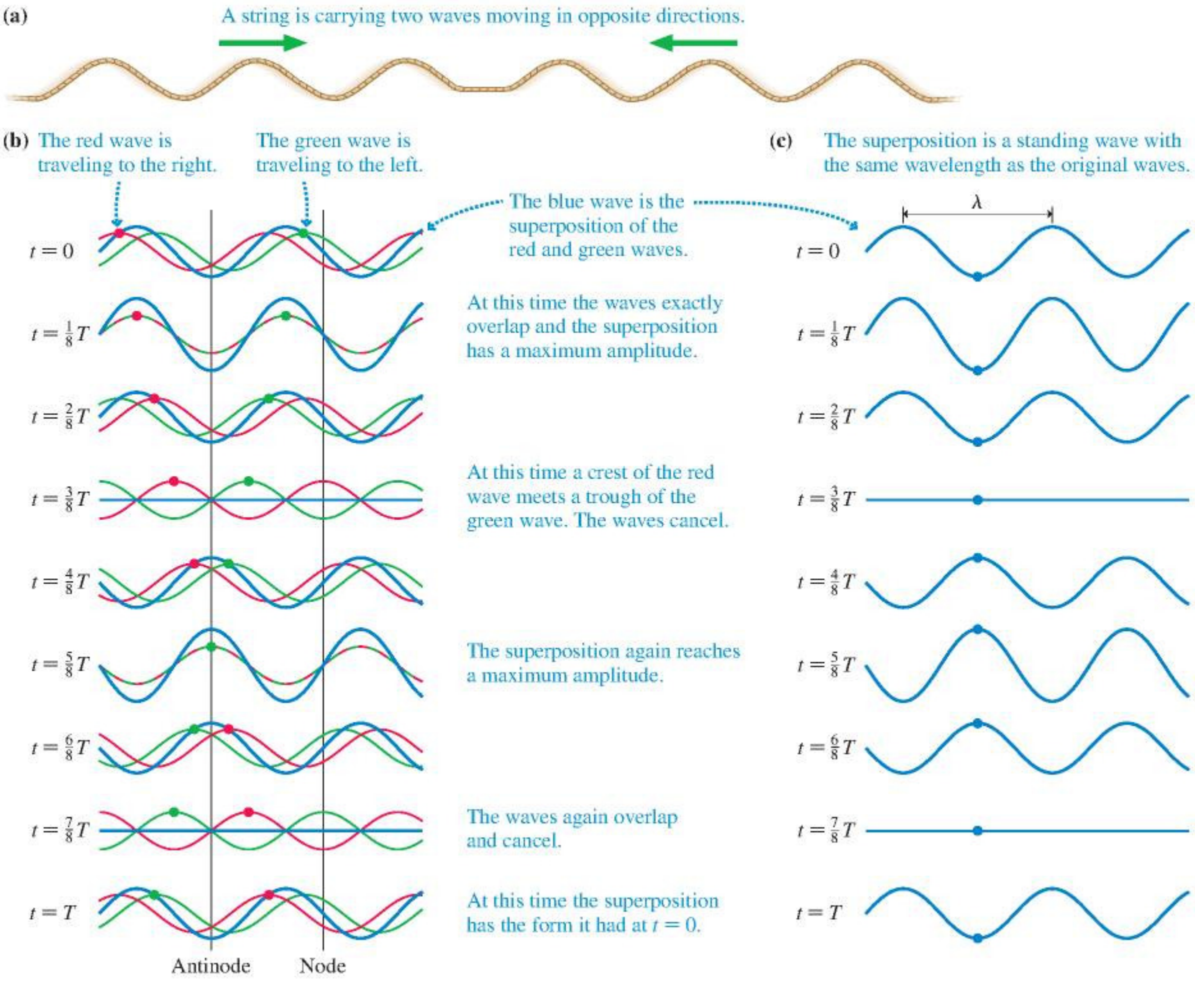


FIGURE 17.4 The superposition of two sinusoidal waves traveling in opposite directions.

A string is carrying two waves moving in opposite directions.

The red wave is traveling to the right.

The green wave is traveling to the left.

The blue wave is the superposition of the red and green waves.

At this time the waves exactly overlap and the superposition has a maximum amplitude.

At this time a crest of the red wave meets a trough of the green wave. The waves cancel.

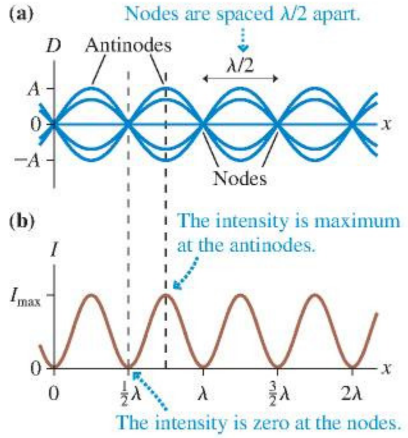
The superposition again reaches a maximum amplitude.

The waves again overlap and cancel.

At this time the superposition has the form it had at $t = 0$.

The superposition is a standing wave with the same wavelength as the original waves.

FIGURE 17.5 The intensity of a standing wave is maximum at the antinodes, zero at the nodes.



Nodes and Antinodes

FIGURE 17.5a has collapsed the nine graphs of Figure 17.4b into a single graphical representation of a standing wave. Compare this to the Figure 17.3 photograph of a vibrating string. A striking feature of a standing-wave pattern is the existence of **nodes,** points that *never move!* **The nodes are spaced λ/2 apart.** Halfway between the nodes are the points where the particles in the medium oscillate with maximum displacement. These points of maximum amplitude are called **antinodes,** and you can see that they are also spaced λ/2 apart.

It seems surprising and counterintuitive that some particles in the medium have no motion at all. To understand this, look closely at the two traveling waves in Figure 17.4a. You will see that the nodes occur at points where at *every instant* of time the displacements of the two traveling waves have equal magnitudes but *opposite signs.* That is, nodes are points of destructive interference where the net displacement is always zero. In contrast, antinodes are points of constructive interference where two displacements of the same sign always add to give a net displacement larger than that of the individual waves.

In Chapter 16 you learned that the *intensity* of a wave is proportional to the square of the amplitude: $I \propto A^2$. You can see in **FIGURE 17.5b** that maximum intensity occurs at the antinodes and that the intensity is zero at the nodes. If this is a sound wave, the loudness is maximum at the antinodes and zero at the nodes. A standing light wave is bright at the antinodes, dark at the nodes. The key idea is that **the intensity is maximum at points of constructive interference and zero at points of destructive interference.**

This photograph shows the Tacoma Narrows suspension bridge on the day in 1940 when it experienced a wind-induced standing-wave oscillation that led to its collapse. The red line shows the original line of the deck of the bridge. You can clearly see the large amplitude of the oscillation and the node at the center of the span.

The Mathematics of Standing Waves

A sinusoidal wave traveling to the right along the x-axis with angular frequency $\omega = 2\pi f$, wave number $k = 2\pi/\lambda$, and amplitude a is

$$D_R = a\sin(kx - \omega t) \tag{17.2}$$

An equivalent wave traveling to the left is

$$D_L = a\sin(kx + \omega t) \tag{17.3}$$

We previously used the symbol A for the wave amplitude, but here we will use a lowercase a to represent the amplitude of each individual wave and reserve A for the amplitude of the net wave.

According to the principle of superposition, the net displacement of the medium when both waves are present is the sum of D_R and D_L:

$$D(x, t) = D_R + D_L = a\sin(kx - \omega t) + a\sin(kx + \omega t) \tag{17.4}$$

We can simplify Equation 17.4 by using the trigonometric identity

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

Doing so gives

$$\begin{aligned} D(x, t) &= a(\sin kx\cos\omega t - \cos kx\sin\omega t) + a(\sin kx\cos\omega t + \cos kx\sin\omega t) \\ &= (2a\sin kx)\cos\omega t \end{aligned} \tag{17.5}$$

It is useful to write Equation 17.5 as

$$D(x, t) = A(x)\cos\omega t \tag{17.6}$$

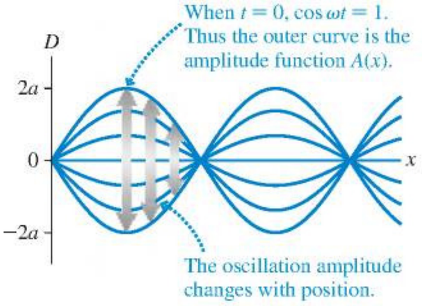
where the **amplitude function** $A(x)$ is defined as

$$A(x) = 2a\sin kx \tag{17.7}$$

The amplitude reaches a maximum value $A_{\max} = 2a$ at points where $\sin kx = 1$.

The displacement $D(x, t)$ given by Equation 17.6 is neither a function of $x - vt$ nor a function of $x + vt$; hence it is *not* a traveling wave. Instead, the $\cos\omega t$ term in

Equation 17.6 describes a medium in which each point oscillates in simple harmonic motion with frequency $f = \omega/2\pi$. The function $A(x) = 2a\sin kx$ gives the amplitude of the oscillation for a particle at position x.



When $t = 0$, $\cos\omega t = 1$. Thus the outer curve is the amplitude function $A(x)$.

The oscillation amplitude changes with position.

FIGURE 17.6 The net displacement resulting from two counter-propagating sinusoidal waves.

FIGURE 17.6 graphs Equation 17.6 at several different instants of time. Notice that the graphs are identical to those of Figure 17.5a, showing us that Equation 17.6 is the mathematical description of a standing wave.

The nodes of the standing wave are the points at which the amplitude is zero. They are located at positions x for which

$$A(x) = 2a\sin kx = 0 \tag{17.8}$$

The sine function is zero if the angle is an integer multiple of π rad, so Equation 17.8 is satisfied if

$$kx_m = \frac{2\pi x_m}{\lambda} = m\pi \qquad m = 0, 1, 2, 3, \ldots \tag{17.9}$$

Thus the position x_m of the mth node is

$$x_m = m\frac{\lambda}{2} \qquad m = 0, 1, 2, 3, \ldots \tag{17.10}$$

You can see that the spacing between two adjacent nodes is $\lambda/2$, in agreement with Figure 17.5b. The nodes are *not* spaced by λ, as you might have expected.

EXAMPLE 17.1 Node spacing on a string

A very long string has a linear density of 5.0 g/m and is stretched with a tension of 8.0 N. 100 Hz waves with amplitudes of 2.0 mm are generated at the ends of the string.

a. What is the node spacing along the resulting standing wave?

b. What is the maximum displacement of the string?

MODEL Two counter-propagating waves of equal frequency create a standing wave.

VISUALIZE The standing wave will look like Figure 17.5a.

SOLVE a. The speed of the waves on the string is

$$v = \sqrt{\frac{T_s}{\mu}} = \sqrt{\frac{8.0\ \text{N}}{0.0050\ \text{kg/m}}} = 40\ \text{m/s}$$

and the wavelength is

$$\lambda = \frac{v}{f} = \frac{40\ \text{m/s}}{100\ \text{Hz}} = 0.40\ \text{m} = 40\ \text{cm}$$

Thus the spacing between adjacent nodes is $\lambda/2 = 20$ cm.

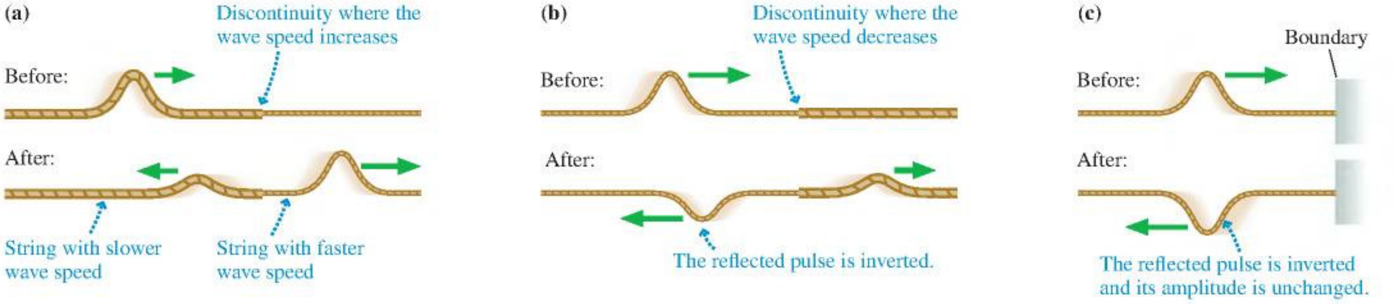
b. The maximum displacement is $A_{\text{max}} = 2a = 4.0$ mm.

17.3 Standing Waves on a String

Wiggling both ends of a very long string is not a practical way to generate standing waves. Instead, as in the photograph in Figure 17.3, standing waves are usually seen on a string that is fixed at both ends. To understand why this condition causes standing waves, we need to examine what happens when a traveling wave encounters a discontinuity.

FIGURE 17.7a shows a *discontinuity* between a string with a larger linear density and one with a smaller linear density. The tension is the same in both strings, so the wave speed is slower on the left, faster on the right. Whenever a wave encounters a discontinuity, some of the wave's energy is *transmitted* forward and some is *reflected.*

FIGURE 17.7 A wave reflects when it encounters a discontinuity or a boundary.



(a) Discontinuity where the wave speed increases

String with slower wave speed

String with faster wave speed

(b) Discontinuity where the wave speed decreases

The reflected pulse is inverted.

(c) The reflected pulse is inverted and its amplitude is unchanged.

Light waves exhibit an analogous behavior when they encounter a piece of glass. Most of the light wave's energy is transmitted through the glass, which is why glass is transparent, but a small amount of energy is reflected. That is how you see your reflection dimly in a storefront window.

In **FIGURE 17.7b**, an incident wave encounters a discontinuity at which the wave speed decreases. In this case, the reflected pulse is *inverted*. A positive displacement of the incident wave becomes a negative displacement of the reflected wave. Because $\sin(\phi + \pi) = -\sin\phi$, we say that the reflected wave has a *phase change of* π *upon reflection*. This aspect of reflection will be important later in the chapter when we look at the interference of light waves.

The wave in **FIGURE 17.7c** reflects from a *boundary*. This is like Figure 17.7b in the limit that the string on the right becomes infinitely massive. Thus the reflection in Figure 17.7c looks like that of Figure 17.7b with one exception: Because there is no transmitted wave, *all* the wave's energy is reflected. Hence **the amplitude of a wave reflected from a boundary is unchanged.**

Creating Standing Waves

FIGURE 17.8 shows a string of length L tied at $x = 0$ and $x = L$. If you wiggle the string in the middle, sinusoidal waves travel outward in both directions and soon reach the boundaries. Because the speed of a reflected wave does not change, **the wavelength and frequency of a reflected sinusoidal wave are unchanged.** Consequently, reflections at the ends of the string cause two waves of *equal amplitude and wavelength* to travel in opposite directions along the string. As we've just seen, these are the conditions that cause a standing wave!

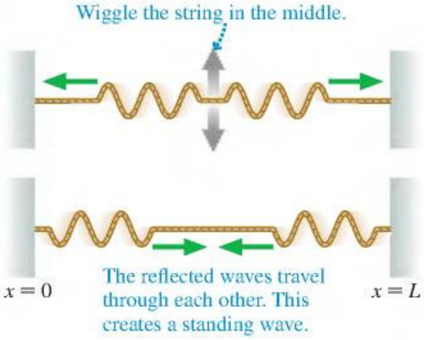


FIGURE 17.8 Reflections at the two boundaries cause a standing wave on the string.

To connect the mathematical analysis of standing waves in Section 17.2 with the physical reality of a string tied down at the ends, we need to impose *boundary conditions*. A **boundary condition** is a mathematical statement of any constraint that *must* be obeyed at the boundary or edge of a medium. Because the string is tied down at the ends, the displacements at $x = 0$ and $x = L$ must be zero at all times. Thus the standing-wave boundary conditions are $D(x = 0, t) = 0$ and $D(x = L, t) = 0$. Stated another way, we require nodes at both ends of the string.

We found that the displacement of a standing wave is $D(x, t) = (2a\sin kx)\cos\omega t$. This equation already satisfies the boundary condition $D(x = 0, t) = 0$. That is, the origin has already been located at a node. The second boundary condition, at $x = L$, requires $D(x = L, t) = 0$. This condition will be met at all times if

$$2a\sin kL = 0 \qquad \text{(boundary condition at } x = L\text{)} \tag{17.11}$$

Equation 17.11 will be true if $\sin kL = 0$, which in turn requires

$$kL = \frac{2\pi L}{\lambda} = m\pi \qquad m = 1, 2, 3, 4, \ldots \tag{17.12}$$

kL must be a multiple of $m\pi$, but $m = 0$ is excluded because L can't be zero.

For a string of fixed length L, the only quantity in Equation 17.12 that can vary is λ. That is, the boundary condition is satisfied only if the wavelength has one of the values

$$\lambda_m = \frac{2L}{m} \qquad m = 1, 2, 3, 4, \ldots \tag{17.13}$$

A standing wave can exist on the string *only* if its wavelength is one of the values given by Equation 17.13. The mth possible wavelength $\lambda_m = 2L/m$ is just the right size so that its mth node is located at the end of the string (at $x = L$).

> **NOTE** Other wavelengths, which would be perfectly acceptable wavelengths for a traveling wave, cannot exist as a *standing* wave of length L because they cannot meet the boundary conditions requiring a node at each end of the string.

If standing waves are possible only for certain wavelengths, then only a few specific oscillation frequencies are allowed. Because $\lambda f = v$ for a sinusoidal wave, the oscillation frequency corresponding to wavelength λ_m is

$$f_m = \frac{v}{\lambda_m} = \frac{v}{2L/m} = m\frac{v}{2L} \qquad m = 1, 2, 3, 4, \ldots \qquad (17.14)$$

The lowest allowed frequency

$$f_1 = \frac{v}{2L} \qquad \text{(fundamental frequency)} \qquad (17.15)$$

which corresponds to wavelength $\lambda_1 = 2L$, is called the **fundamental frequency** of the string. The allowed frequencies can be written in terms of the fundamental frequency as

$$f_m = mf_1 \qquad m = 1, 2, 3, 4, \ldots \qquad (17.16)$$

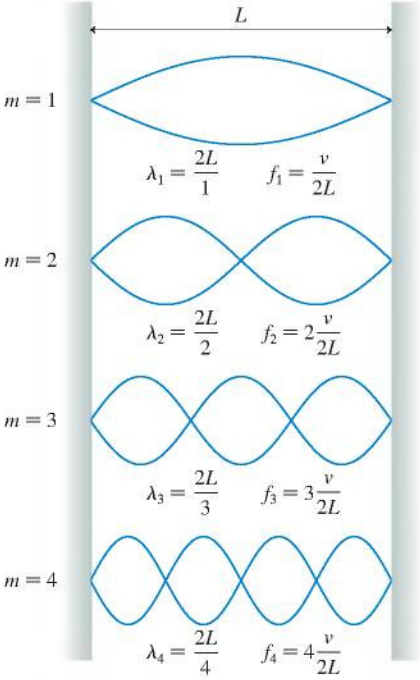
The allowed standing-wave frequencies are all integer multiples of the fundamental frequency. The higher-frequency standing waves are called **harmonics,** with the $m = 2$ wave at frequency f_2 called the *second harmonic,* the $m = 3$ wave called the *third harmonic,* and so on.

FIGURE 17.9 graphs the first four possible standing waves on a string of fixed length L. These possible standing waves are called the **modes** of the string, or sometimes the *normal modes.* Each mode, numbered by the integer m, has a unique wavelength and frequency. Keep in mind that these drawings simply show the *envelope,* or outer edge, of the oscillations. The string is continuously oscillating at all positions between these edges, as we showed in more detail in Figure 17.5a.

There are three things to note about the modes of a string.

1. m is the number of *antinodes* on the standing wave, not the number of nodes. You can tell a string's mode of oscillation by counting the number of antinodes.
2. The *fundamental mode,* with $m = 1$, has $\lambda_1 = 2L$, not $\lambda_1 = L$. Only half of a wavelength is contained between the boundaries, a direct consequence of the fact that the spacing between nodes is $\lambda/2$.
3. The frequencies of the normal modes form a series: $f_1, 2f_1, 3f_1, 4f_1, \ldots$. The fundamental frequency f_1 can be found as the *difference* between the frequencies of any two adjacent modes. That is, $f_1 = \Delta f = f_{m+1} - f_m$.

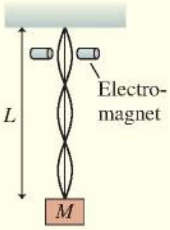
FIGURE 17.9 The first four modes for standing waves on a string of length L.



EXAMPLE 17.2 Measuring *g*

Standing-wave frequencies can be measured very accurately. Consequently, standing waves are often used in experiments to make accurate measurements of other quantities. One such experiment, shown in **FIGURE 17.10**, uses standing waves to measure the free-fall acceleration g. A heavy mass is suspended from a 1.65-m-long, 5.85 g steel wire; then an oscillating magnetic field (because steel is magnetic) is used to excite the $m = 3$ standing wave on the wire. Measuring the frequency for different masses yields the data given in the table. Analyze these data to determine the local value of g.

FIGURE 17.10 An experiment to measure g.



Mass (kg)	f_3 (Hz)
2.00	68
4.00	97
6.00	117
8.00	135
10.00	152

MODEL The hanging mass creates tension in the wire. This establishes the wave speed along the wire and thus the frequencies of standing waves. Masses of a few kg might stretch the wire a mm or so, but that doesn't change the length L until the third decimal place. The mass of the wire itself is insignificant in comparison to that of the hanging mass. We'll be justified in determining g to three significant figures.

SOLVE The frequency of the third harmonic is

$$f_3 = \frac{3}{2}\frac{v}{L}$$

The wave speed on the wire is

$$v = \sqrt{\frac{T_s}{\mu}} = \sqrt{\frac{Mg}{m/L}} = \sqrt{\frac{MgL}{m}}$$

where Mg is the weight of the hanging mass, and thus the tension in the wire, while m is the mass of the wire. Combining these two equations, we have

Continued

$$f_3 = \frac{3}{2}\sqrt{\frac{Mg}{mL}} = \frac{3}{2}\sqrt{\frac{g}{mL}}\sqrt{M}$$

Squaring both sides gives

$$f_3^2 = \frac{9g}{4mL}M$$

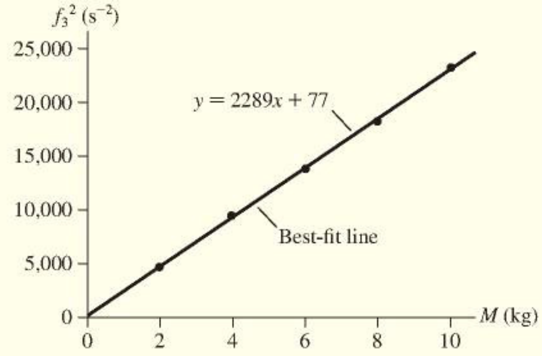
A graph of the square of the standing-wave frequency versus mass M should be a straight line passing through the origin with slope $9g/4mL$. We can use the experimental slope to determine g.

FIGURE 17.11 is a graph of f_3^2 versus M. The slope of the best-fit line is 2289 $\text{kg}^{-1}\text{s}^{-2}$, from which we find

$$g = \text{slope} \times \frac{4mL}{9}$$

$$= 2289\ \text{kg}^{-1}\text{s}^{-2} \times \frac{4(0.00585\ \text{kg})(1.65\ \text{m})}{9} = 9.82\ \text{m/s}^2$$

FIGURE 17.11 Graph of the data.



ASSESS The fact that the graph is linear and passes through the origin confirms our model. This is an important reason for having multiple data points rather than using only one mass.

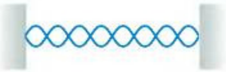
STOP TO THINK 17.2 A standing wave on a string vibrates as shown at the right. Suppose the string tension is quadrupled while the frequency and the length of the string are held constant. Which standing-wave pattern is produced?

Original standing wave

(a)

(b)

(c)

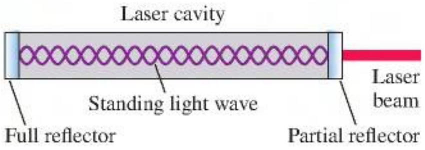


(d)

Standing Electromagnetic Waves

Because electromagnetic waves are transverse waves, a standing electromagnetic wave is very much like a standing wave on a string. Standing electromagnetic waves can be established between two parallel mirrors that reflect light back and forth. The mirrors are boundaries, analogous to the boundaries at the ends of a string. In fact, this is exactly how a laser operates. The two facing mirrors in **FIGURE 17.12** form what is called a *laser cavity.*

FIGURE 17.12 A laser contains a standing light wave between two parallel mirrors.



Because the mirrors act like the points to which a string is tied, the light wave must have a node at the surface of each mirror. One of the mirrors is only partially reflective, to allow some light to escape and form the laser beam, but this doesn't affect the boundary condition.

Because the boundary conditions are the same, Equations 17.13 and 17.14 for λ_m and f_m apply to a laser just as they do to a vibrating string. The primary difference is the size of the wavelength. A typical laser cavity has a length $L \approx 30$ cm, and visible light has a wavelength $\lambda \approx 600$ nm. The standing light wave in a laser cavity has a mode number m that is approximately

$$m = \frac{2L}{\lambda} \approx \frac{2 \times 0.30\ \text{m}}{6.00 \times 10^{-7}\ \text{m}} = 1{,}000{,}000$$

In other words, the standing light wave inside a laser cavity has approximately one million antinodes! This is a consequence of the very short wavelength of light.

EXAMPLE 17.3 The standing light wave inside a laser

Helium-neon lasers emit the red laser light commonly used in classroom demonstrations and supermarket checkout scanners. A helium-neon laser operates at a wavelength of precisely 632.9924 nm when the spacing between the mirrors is 310.372 mm.

a. In which mode does this laser operate?

b. What is the next longest wavelength that could form a standing wave in this laser cavity?

MODEL The light wave forms a standing wave between the two mirrors.

VISUALIZE The standing wave looks like Figure 17.12.

SOLVE a. We can use $\lambda_m = 2L/m$ to find that m (the mode) is

$$m = \frac{2L}{\lambda_m} = \frac{2(0.310372\text{ m})}{6.329924 \times 10^{-7}\text{ m}} = 980{,}650$$

There are 980,650 antinodes in the standing light wave.

b. The next longest wavelength that can fit in this laser cavity will have one fewer node. It will be the $m = 980{,}649$ mode and its wavelength will be

$$\lambda = \frac{2L}{m} = \frac{2(0.310372\text{ m})}{980{,}649} = 632.9930\text{ nm}$$

ASSESS The wavelength increases by a mere 0.0006 nm when the mode number is decreased by 1.

Microwaves, with a wavelength of a few centimeters, can also set up standing waves. This is not always good. If the microwaves in a microwave oven form a standing wave, there are nodes where the electromagnetic field intensity is always zero. These nodes cause cold spots where the food does not heat. Although designers of microwave ovens try to prevent standing waves, ovens usually do have cold spots spaced $\lambda/2$ apart at nodes in the microwave field. A turntable in a microwave oven keeps the food moving so that no part of your dinner remains at a node.

17.4 Standing Sound Waves and Musical Acoustics

A long, narrow column of air, such as the air in a tube or pipe, can support a *longitudinal* standing sound wave. Longitudinal waves are somewhat trickier than string waves because a graph—showing displacement *parallel* to the tube—is not a picture of the wave.

To illustrate the ideas, **FIGURE 17.13** is a series of three graphs and pictures that show the $m = 2$ standing wave inside a column of air closed at both ends. We call this a *closed-closed tube*. The air at the closed ends cannot oscillate because the air molecules are pressed up against the wall, unable to move; hence **a closed end of a column of air must be a displacement node.** Thus the boundary conditions—nodes at the ends—are the same as for a standing wave on a string.

FIGURE 17.13 The $m = 2$ standing sound wave in a closed-closed tube of air.

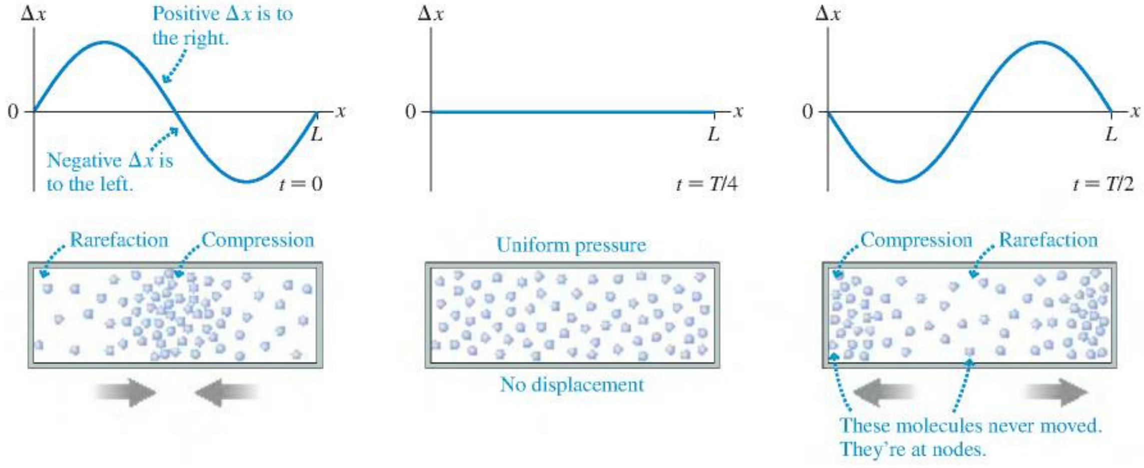
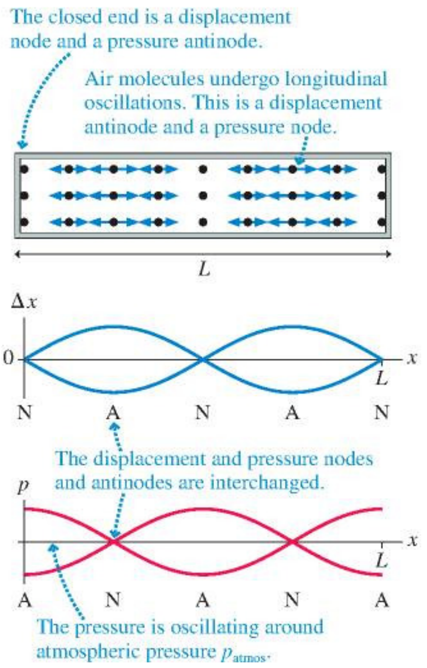


FIGURE 17.14 The $m = 2$ longitudinal standing wave can be represented as a displacement wave or as a pressure wave.



Although the graph looks familiar, it is now a graph of *longitudinal* displacement. At $t = 0$, positive displacements in the left half and negative displacements in the right half cause all the air molecules to converge at the center of the tube. The density and pressure rise at the center and fall at the ends—a *compression* and *rarefaction* in the terminology of Chapter 16. A half cycle later, the molecules have rushed to the ends of the tube. Now the pressure is maximum at the ends, minimum in the center. Try to visualize the air molecules sloshing back and forth this way.

FIGURE 17.14 combines these illustrations into a single picture showing where the molecules are oscillating (antinodes) and where they're not (nodes). A graph of the displacement Δx looks just like the $m = 2$ graph of a standing wave on a string. Because the boundary conditions are the same, the possible wavelengths and frequencies of standing waves in a closed-closed tube are the same as for a string of the same length.

It is often useful to think of sound as a *pressure wave* rather than a displacement wave, and the bottom graph in Figure 17.14 shows the $m = 2$ pressure standing wave in a closed-closed tube. Notice that the pressure is oscillating around p_{atmos}, its equilibrium value.

We showed in « Section 16.6 that the pressure in a sound wave is minimum or maximum at points where the displacement is zero, and vice versa. Consequently, **the nodes and antinodes of the pressure wave are interchanged with those of the displacement wave.** You can see in Figure 17.13 that the gas molecules are alternately pushed up against the ends of the tube, then pulled away, causing the pressure at the closed ends to oscillate with maximum amplitude—an antinode—at a point where the displacement is a node.

EXAMPLE 17.4 Singing in the shower

A shower stall is 2.45 m (8 ft) tall. For what frequencies less than 500 Hz are there standing sound waves in the shower stall?

MODEL The shower stall, to a first approximation, is a column of air 2.45 m long. It is closed at the ends by the ceiling and floor. Assume a 20°C speed of sound.

VISUALIZE A standing sound wave will have nodes at the ceiling and the floor. The $m = 2$ mode will look like Figure 17.14 rotated 90°.

SOLVE The fundamental frequency for a standing sound wave in this air column is

$$f_1 = \frac{v}{2L} = \frac{343\text{ m/s}}{2(2.45\text{ m})} = 70\text{ Hz}$$

The possible standing-wave frequencies are integer multiples of the fundamental frequency. These are 70 Hz, 140 Hz, 210 Hz, 280 Hz, 350 Hz, 420 Hz, and 490 Hz.

ASSESS The many possible standing waves in a shower cause the sound to *resonate*, which helps explain why some people like to sing in the shower. Our approximation of the shower stall as a one-dimensional tube is actually a bit too simplistic. A three-dimensional analysis would find additional modes, making the "sound spectrum" even richer.

Tubes with Openings

Air columns closed at both ends are of limited interest unless, as in Example 17.4, you are inside the column. Columns of air that *emit* sound are open at one or both ends. Many musical instruments fit this description. For example, a flute is a tube of air open at both ends. The flutist blows across one end to create a standing wave inside the tube, and a note of that frequency is emitted from both ends of the flute. (The blown end of a flute is open on the side, rather than across the tube. That is necessary for practical reasons of how flutes are played, but from a physics perspective this is the "end" of the tube because it opens the tube to the atmosphere.) A trumpet, however, is open at the bell end but is *closed* by the player's lips at the other end.

You saw earlier that a wave is partially transmitted and partially reflected at a discontinuity. When a sound wave traveling through a tube of air reaches an open end, some of the wave's energy is transmitted out of the tube to become the sound that you hear and some portion of the wave is reflected back into the tube. These reflections, analogous to the reflection of a string wave from a boundary, allow standing sound waves to exist in a tube of air that is open at one or both ends.

Not surprisingly, the *boundary condition* at the open end of a column of air is not the same as the boundary condition at a closed end. The air pressure at the open end of a tube is constrained to match the atmospheric pressure of the surrounding air. Consequently, the open end of a tube must be a pressure node. Because pressure nodes and antinodes are interchanged with those of the displacement wave, **an open end of an air column is required to be a displacement antinode.** (A careful analysis shows that the antinode is actually just outside the open end, but for our purposes we'll assume the antinode is exactly at the open end.)

FIGURE 17.15 The first three standing sound wave modes in columns of air with different boundary conditions.

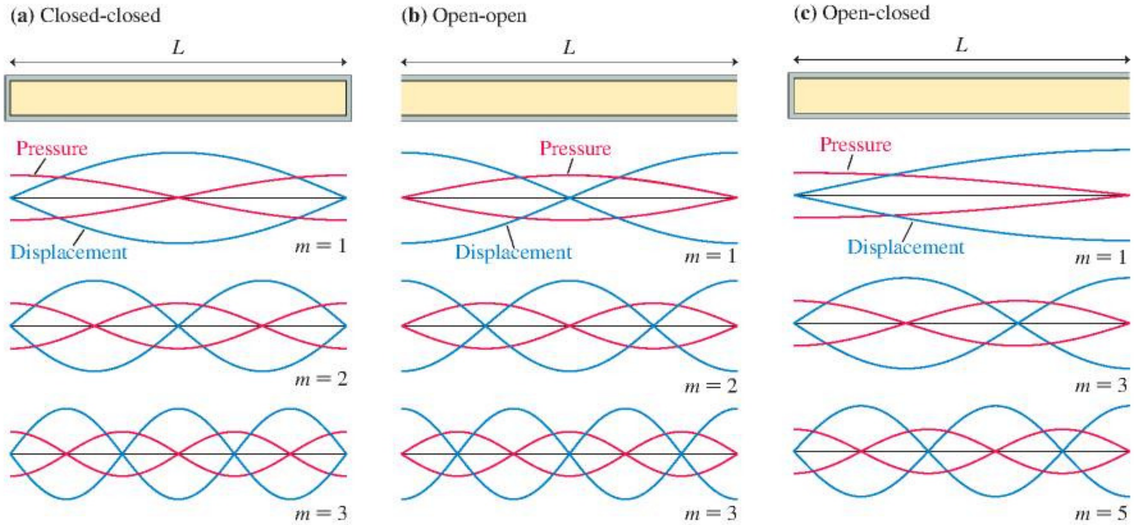


FIGURE 17.15 shows displacement and pressure graphs of the first three standing-wave modes of a tube closed at both ends (a *closed-closed tube*), a tube open at both ends (an *open-open tube*), and a tube open at one end but closed at the other (an *open-closed tube*), all with the same length L. Notice the pressure and displacement boundary conditions. The standing wave in the open-open tube looks like the closed-closed tube except that the positions of the nodes and antinodes are interchanged. In both cases there are m half-wavelength segments between the ends; thus the wavelengths and frequencies of an open-open tube and a closed-closed tube are the same as those of a string tied at both ends:

$$\lambda_m = \frac{2L}{m} \qquad m = 1, 2, 3, 4, \ldots$$
$$f_m = m\frac{v}{2L} = mf_1 \qquad \text{(open-open or closed-closed tube)} \qquad (17.17)$$

The open-closed tube is different. The fundamental mode has only one-quarter of a wavelength in a tube of length L; hence the $m = 1$ wavelength is $\lambda_1 = 4L$. This is twice the λ_1 wavelength of an open-open or a closed-closed tube. Consequently, **the fundamental frequency of an open-closed tube is half that of an open-open or a closed-closed tube of the same length.** It will be left as a homework problem for you to show that the possible wavelengths and frequencies of an open-closed tube of length L are

$$\lambda_m = \frac{4L}{m} \qquad m = 1, 3, 5, 7, \ldots$$
$$f_m = m\frac{v}{4L} = mf_1 \qquad \text{(open-closed tube)} \qquad (17.18)$$

Notice that m in Equation 17.18 takes on only *odd* values.

EXAMPLE 17.5 Resonances of the ear canal

The eardrum, which transmits sound vibrations to the sensory organs of the inner ear, lies at the end of the ear canal. For adults, the ear canal is about 2.5 cm in length. What frequency standing waves can occur in the ear canal that are within the range of human hearing? The speed of sound in the warm air of the ear canal is 350 m/s.

MODEL The ear canal is open to the air at one end, closed by the eardrum at the other. We can model it as an open-closed tube. The standing waves will be those of Figure 17.15c.

SOLVE The lowest standing-wave frequency is the fundamental frequency for a 2.5-cm-long open-closed tube:

$$f_1 = \frac{v}{4L} = \frac{350 \text{ m/s}}{4(0.025 \text{ m})} = 3500 \text{ Hz}$$

Standing waves also occur at the harmonics, but an open-closed tube has only odd harmonics. These are

$$f_3 = 3f_1 = 10{,}500 \text{ Hz}$$
$$f_5 = 5f_1 = 17{,}500 \text{ Hz}$$

Higher harmonics are beyond the range of human hearing, as discussed in Section 16.5.

ASSESS The ear canal is short, so we expected the standing-wave frequencies to be relatively high. The air in your ear canal responds readily to sounds at these frequencies—what we call a *resonance* of the ear canal—and transmits theses sounds to the eardrum. Consequently, your ear actually is slightly more sensitive to sounds with frequencies around 3500 Hz and 10,500 Hz than to sounds at nearby frequencies.

STOP TO THINK 17.3 An open-open tube of air supports standing waves at frequencies of 300 Hz and 400 Hz and at no frequencies between these two. The second harmonic of this tube has frequency

a. 100 Hz b. 200 Hz c. 400 Hz d. 600 Hz e. 800 Hz

Musical Instruments

An important application of standing waves is to musical instruments. Instruments such as the guitar, the piano, and the violin have strings fixed at the ends and tightened to create tension. A disturbance generated on the string by plucking, striking, or bowing it creates a standing wave on the string.

The fundamental frequency of a vibrating string is

$$f_1 = \frac{v}{2L} = \frac{1}{2L}\sqrt{\frac{T_s}{\mu}}$$

where T_s is the tension in the string and μ is its linear density. The fundamental frequency is the musical note you hear when the string is sounded. Increasing the tension in the string raises the fundamental frequency, which is how stringed instruments are tuned.

NOTE v is the wave speed *on the string,* not the speed of sound in air.

For the guitar or the violin, the strings are all the same length and under approximately the same tension. Were that not the case, the neck of the instrument would tend to twist toward the side of higher tension. The strings have different frequencies because they differ in linear density: The lower-pitched strings are "fat" while the higher-pitched strings are "skinny." This difference changes the frequency by changing the wave speed. *Small* adjustments are then made in the tension to bring each string to the exact desired frequency. Once the instrument is tuned, you play it by using your fingertips to alter the effective length of the string. As you shorten the string's length, the frequency and pitch go up.

A piano covers a much wider range of frequencies than a guitar or violin. This range cannot be produced by changing only the linear densities of the strings. The high end would have strings too thin to use without breaking, and the low end would have solid rods rather than flexible wires! So a piano is tuned through a combination of changing the linear density *and* the length of the strings. The bass note strings are not only fatter, they are also longer.

The strings on a harp vibrate as standing waves. Their frequencies determine the notes that you hear.

With a wind instrument, blowing into the mouthpiece creates a standing sound wave inside a tube of air. The player changes the notes by using her fingers to cover holes or open valves, changing the length of the tube and thus its frequency. The fact that the holes are on the side makes very little difference; the first open hole becomes an antinode because the air is free to oscillate in and out of the opening.

A wind instrument's frequency depends on the speed of sound *inside* the instrument. But the speed of sound depends on the temperature of the air. When a wind player first blows into the instrument, the air inside starts to rise in temperature. This increases the sound speed, which in turn raises the instrument's frequency for each note until the air temperature reaches a steady state. Consequently, wind players must "warm up" before tuning their instrument.

Many wind instruments have a "buzzer" at one end of the tube, such as a vibrating reed on a saxophone or vibrating lips on a trombone. Buzzers generate a continuous range of frequencies rather than single notes, which is why they sound like a "squawk" if you play on just the mouthpiece without the rest of the instrument. When a buzzer is connected to the body of the instrument, most of those frequencies cause no response of the air molecules. But the frequency from the buzzer that matches the fundamental frequency of the instrument causes the buildup of a large-amplitude response at just that frequency—a standing-wave resonance. This is the energy input that generates and sustains the musical note.

EXAMPLE 17.6 Flutes and clarinets

A clarinet is 66.0 cm long. A flute is nearly the same length, with 63.6 cm between the hole the player blows across and the end of the flute. What are the frequencies of the lowest note and the next higher harmonic on a flute and on a clarinet? The speed of sound in warm air is 350 m/s.

MODEL The flute is an open-open tube, open at the end as well as at the hole the player blows across. A clarinet is an open-closed tube because the player's lips and the reed seal the tube at the upper end.

SOLVE The lowest frequency is the fundamental frequency. For the flute, an open-open tube, this is

$$f_1 = \frac{v}{2L} = \frac{350\text{ m/s}}{2(0.636\text{ m})} = 275\text{ Hz}$$

The clarinet, an open-closed tube, has

$$f_1 = \frac{v}{4L} = \frac{350\text{ m/s}}{4(0.660\text{ m})} = 133\text{ Hz}$$

The next higher harmonic on the flute's open-open tube is $m = 2$ with frequency $f_2 = 2f_1 = 550$ Hz. An open-closed tube has only odd harmonics, so the next higher harmonic of the clarinet is $f_3 = 3f_1 = 399$ Hz.

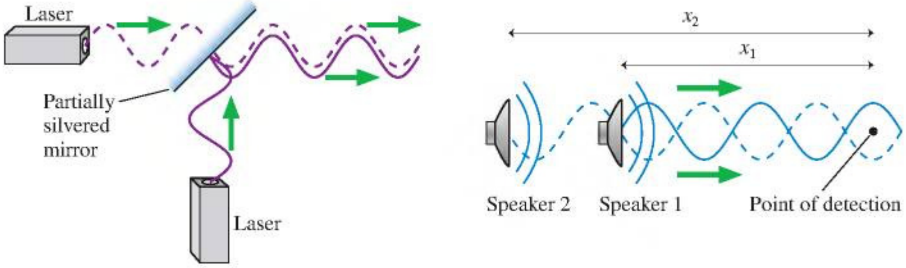
ASSESS The clarinet plays a much lower note than the flute—musically, about an octave lower—because it is an open-closed tube. It's worth noting that neither of our fundamental frequencies is exactly correct because our open-open and open-closed tube models are a bit too simplified to adequately describe a real instrument. However, both calculated frequencies are close because our models do capture the essence of the physics.

A vibrating string plays the musical note corresponding to the fundamental frequency f_1, so stringed instruments must use several strings to obtain a reasonable range of notes. In contrast, wind instruments can sound at the second or third harmonic of the tube of air (f_2 or f_3). These higher frequencies are sounded by *overblowing* (flutes, brass instruments) or with keys that open small holes to encourage the formation of an antinode at that point (clarinets, saxophones). The controlled use of these higher harmonics gives wind instruments a wide range of notes.

17.5 Interference in One Dimension

One of the most basic characteristics of waves is the ability of two waves to combine into a single wave whose displacement is given by the principle of superposition. The pattern resulting from the superposition of two waves is often called **interference.** A standing wave is the interference pattern produced when two waves of equal frequency travel in opposite directions. In this section we will look at the interference of two waves traveling in the *same* direction.

FIGURE 17.16 Two overlapped waves travel along the x-axis.



(a) Two overlapped light waves

(b) Two overlapped sound waves

FIGURE 17.16a shows two light waves impinging on a partially silvered mirror. Such a mirror partially transmits and partially reflects each wave, causing two *overlapped* light waves to travel along the x-axis to the right of the mirror. Or consider the two loudspeakers in **FIGURE 17.16b**. The sound wave from loudspeaker 2 passes just to the side of loudspeaker 1; hence two overlapped sound waves travel to the right along the x-axis. We want to find out what happens when two overlapped waves travel in the same direction along the same axis.

Figure 17.16b shows a point on the x-axis where the overlapped waves are detected, either by your ear or by a microphone. This point is distance x_1 from loudspeaker 1 and distance x_2 from loudspeaker 2. (We will use loudspeakers and sound waves for most of our examples, but our analysis is valid for any wave.) What is the amplitude of the combined waves at this point?

Throughout this section, **we will assume that the waves are sinusoidal, have the same frequency and amplitude, and travel to the right along the x-axis.** Thus we can write the displacements of the two waves as

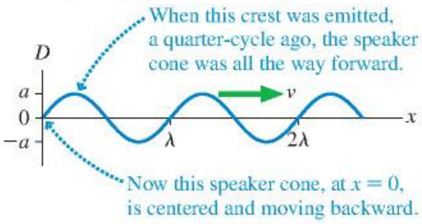
$$\begin{aligned} D_1(x_1, t) &= a\sin(kx_1 - \omega t + \phi_{10}) = a\sin\phi_1 \\ D_2(x_2, t) &= a\sin(kx_2 - \omega t + \phi_{20}) = a\sin\phi_2 \end{aligned} \qquad (17.19)$$

where ϕ_1 and ϕ_2 are the *phases* of the waves. Both waves have the same wave number $k = 2\pi/\lambda$ and the same angular frequency $\omega = 2\pi f$.

The phase constants ϕ_{10} and ϕ_{20} are characteristics of *the sources,* not the medium. **FIGURE 17.17** shows snapshot graphs at $t = 0$ of waves emitted by three sources with phase constants $\phi_0 = 0$ rad, $\phi_0 = \pi/2$ rad, and $\phi_0 = \pi$ rad. You can see that **the phase constant tells us what the source is doing at $t = 0$.** For example, a loudspeaker at its center position and moving backward at $t = 0$ has $\phi_0 = 0$ rad. Looking back at Figure 17.16b, you can see that loudspeaker 1 has phase constant $\phi_{10} = 0$ rad and loudspeaker 2 has $\phi_{20} = \pi$ rad.

FIGURE 17.17 Waves from three sources having phase constants $\phi_0 = 0$ rad, $\phi_0 = \pi/2$ rad, and $\phi_0 = \pi$ rad.

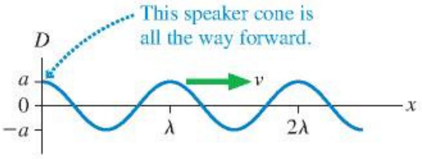
(a) Snapshot graph at $t = 0$ for $\phi_0 = 0$ rad



When this crest was emitted, a quarter-cycle ago, the speaker cone was all the way forward.

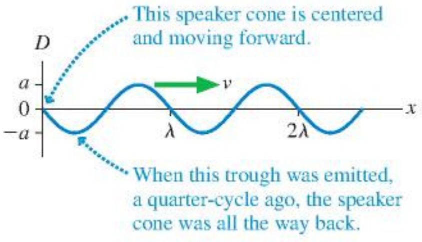
Now this speaker cone, at $x = 0$, is centered and moving backward.

(b) Snapshot graph at $t = 0$ for $\phi_0 = \pi/2$ rad



This speaker cone is all the way forward.

(c) Snapshot graph at $t = 0$ for $\phi_0 = \pi$ rad



This speaker cone is centered and moving forward.

When this trough was emitted, a quarter-cycle ago, the speaker cone was all the way back.

NOTE We will often consider *identical sources,* by which we mean that $\phi_{20} = \phi_{10}$. That is, the sources oscillate in phase.

Let's examine overlapped waves graphically before diving into the mathematics. **FIGURE 17.18** shows two important situations. In part a, the crests of the two waves are aligned as they travel along the x-axis. In part b, the crests of one wave align with the troughs of the other wave. The graphs and the wave fronts are slightly displaced from each other so that you can see what each wave is doing, but the *physical situation* is one in which the waves are traveling *on top of* each other.

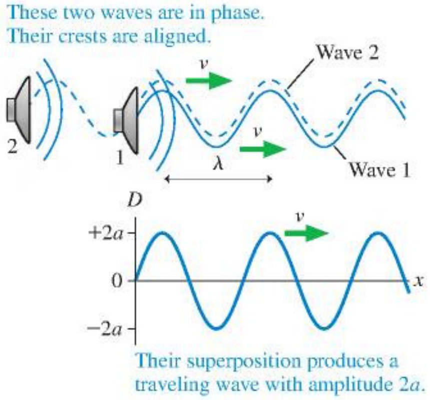
The two waves of Figure 17.18a have the same displacement at every point: $D_1(x) = D_2(x)$. Two waves that are aligned crest to crest and trough to trough are said to be **in phase.** Waves that are in phase march along "in step" with each other.

When we combine two in-phase waves, using the principle of superposition, the net displacement at each point is twice the displacement of each individual wave. The superposition of two waves to create a traveling wave with an amplitude *larger* than either individual wave is called **constructive interference.** When the waves are exactly in phase, giving $A = 2a$, we have *maximum constructive interference.*

In Figure 17.18b, where the crests of one wave align with the troughs of the other, the waves march along "out of step" with $D_1(x) = -D_2(x)$ at every point. Two waves that are aligned crest to trough are said to be *180° out of phase* or, more generally, just **out of phase.** A superposition of two waves to create a wave with an amplitude smaller than either individual wave is called **destructive interference.** In this case, because $D_1 = -D_2$, the net displacement is *zero* at *every point* along the axis. The combination of two waves that cancel each other to give no wave is called *perfect destructive interference.*

FIGURE 17.18 Interference of two waves traveling along the x-axis.

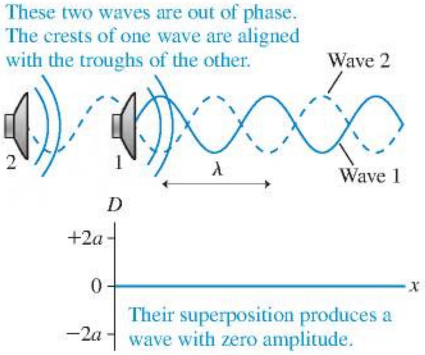
(a) Maximum constructive interference



These two waves are in phase. Their crests are aligned.

Their superposition produces a traveling wave with amplitude $2a$.

(b) Perfect destructive interference



These two waves are out of phase. The crests of one wave are aligned with the troughs of the other.

Their superposition produces a wave with zero amplitude.

NOTE Perfect destructive interference occurs only if the two waves have exactly equal amplitudes, as we're assuming. A 180° phase difference always produces *maximum destructive interference,* but the cancellation won't be perfect if there is any difference in the amplitudes.

The Phase Difference

To understand interference, we need to focus on the *phases* of the two waves, which are

$$\begin{aligned}\phi_1 &= kx_1 - \omega t + \phi_{10}\\ \phi_2 &= kx_2 - \omega t + \phi_{20}\end{aligned} \tag{17.20}$$

The difference between the two phases ϕ_1 and ϕ_2, called the **phase difference** $\Delta\phi$, is

$$\begin{aligned}\Delta\phi = \phi_2 - \phi_1 &= (kx_2 - \omega t + \phi_{20}) - (kx_1 - \omega t + \phi_{10})\\ &= k(x_2 - x_1) + (\phi_{20} - \phi_{10})\\ &= 2\pi\frac{\Delta x}{\lambda} + \Delta\phi_0\end{aligned} \tag{17.21}$$

You can see that there are two contributions to the phase difference. $\Delta x = x_2 - x_1$, the distance between the two sources, is called **path-length difference.** It is the extra distance traveled by wave 2 on the way to the point where the two waves are combined. $\Delta\phi_0 = \phi_{20} - \phi_{10}$ is the *inherent phase difference* between the sources.

The condition of being in phase, where crests are aligned with crests and troughs with troughs, is $\Delta\phi = 0, 2\pi, 4\pi$, or any integer multiple of 2π rad. Thus the condition for maximum constructive interference is

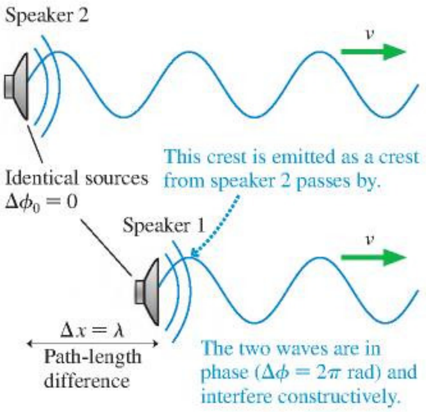
Maximum constructive interference:

$$\Delta\phi = 2\pi\frac{\Delta x}{\lambda} + \Delta\phi_0 = m \cdot 2\pi \text{ rad} \qquad m = 0, 1, 2, 3, \ldots \tag{17.22}$$

For identical sources, which have $\Delta\phi_0 = 0$ rad, maximum constructive interference occurs when $\Delta x = m\lambda$. That is, **two identical sources produce maximum constructive interference when the path-length difference is an integer number of wavelengths.**

FIGURE 17.19 shows two identical sources (i.e., the two loudspeakers are doing the same thing at the same time), so $\Delta\phi_0 = 0$ rad. The path-length difference Δx is the extra distance traveled by the wave from loudspeaker 2 before it combines with loudspeaker 1. In this case, $\Delta x = \lambda$. Because a wave moves forward exactly one wavelength during one period, loudspeaker 1 emits a crest exactly as a crest of wave 2 passes by. The two waves are "in step," with $\Delta\phi = 2\pi$ rad, so the two waves interfere constructively to produce a wave of amplitude $2a$.

FIGURE 17.19 Two identical sources one wavelength apart.



This crest is emitted as a crest from speaker 2 passes by.

The two waves are in phase ($\Delta\phi = 2\pi$ rad) and interfere constructively.

Maximum destructive interference, where the crests of one wave are aligned with the troughs of the other, occurs when two waves are *out of phase,* meaning that $\Delta\phi = \pi$, 3π, 5π, or any odd multiple of π rad. Thus the condition for maximum destructive interference is

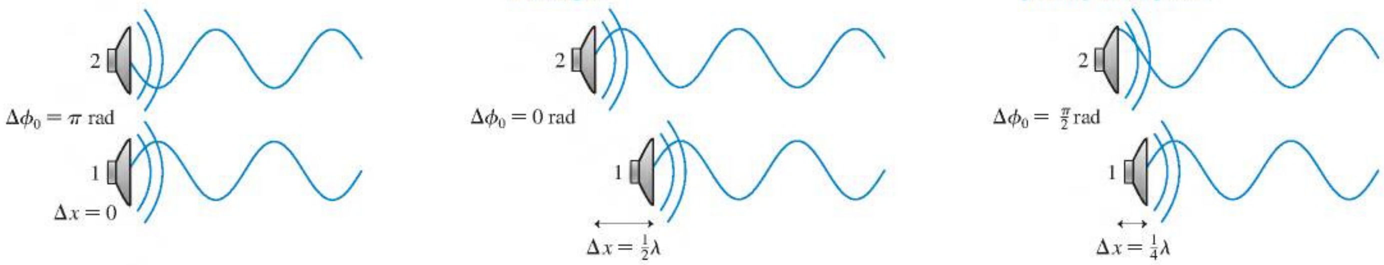
Maximum destructive interference:

$$\Delta\phi = 2\pi\frac{\Delta x}{\lambda} + \Delta\phi_0 = \left(m + \tfrac{1}{2}\right) \cdot 2\pi \text{ rad} \qquad m = 0, 1, 2, 3, \ldots \tag{17.23}$$

For identical sources, which have $\Delta\phi_0 = 0$ rad, maximum destructive interference occurs when $\Delta x = (m + \frac{1}{2})\lambda$. **That is, two identical sources produce maximum destructive interference when the path-length difference is a half-integer number of wavelengths.**

Two waves can be out of phase because the sources are located at different positions, because the sources themselves are out of phase, or because of a combination of these two. FIGURE 17.20 illustrates these ideas by showing three different ways in which two waves interfere destructively. Each of these three arrangements creates waves with $\Delta\phi = \pi$ rad.

FIGURE 17.20 Destructive interference three ways.



(a) The sources are out of phase.

(b) Identical sources are separated by half a wavelength.

(c) The sources are both separated and partially out of phase.

NOTE Don't confuse the phase difference of the waves ($\Delta\phi$) with the phase difference of the sources ($\Delta\phi_0$). It is $\Delta\phi$, the phase difference of the waves, that governs interference.

EXAMPLE 17.7 Interference between two sound waves

You are standing in front of two side-by-side loudspeakers playing sounds of the same frequency. Initially there is almost no sound at all. Then one of the speakers is moved slowly away from you. The sound intensity increases as the separation between the speakers increases, reaching a maximum when the speakers are 0.75 m apart. Then, as the speaker continues to move, the intensity starts to decrease. What is the distance between the speakers when the sound intensity is again a minimum?

MODEL The changing sound intensity is due to the interference of two overlapped sound waves.

VISUALIZE Moving one speaker relative to the other changes the phase difference between the waves.

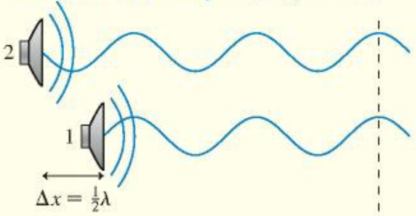
SOLVE A minimum sound intensity implies that the two sound waves are interfering destructively. Initially the loudspeakers are side by side, so the situation is as shown in Figure 17.20a with $\Delta x = 0$ and $\Delta\phi_0 = \pi$ rad. That is, the speakers themselves are out of phase. Moving one of the speakers does not change $\Delta\phi_0$, but it does change the path-length difference Δx and thus increases the overall phase difference $\Delta\phi$. Constructive interference, causing maximum intensity, is reached when

$$\Delta\phi = 2\pi\frac{\Delta x}{\lambda} + \Delta\phi_0 = 2\pi\frac{\Delta x}{\lambda} + \pi = 2\pi \text{ rad}$$

where we used $m = 1$ because this is the first separation giving constructive interference. The speaker separation at which this occurs is $\Delta x = \lambda/2$. This is the situation shown in FIGURE 17.21.

FIGURE 17.21 Two out-of-phase sources generate waves that are in phase if the sources are one half-wavelength apart.

The sources are out of phase, $\Delta\phi_0 = \pi$ rad.



The sources are separated by half a wavelength.

As a result, the waves are in phase.

Because $\Delta x = 0.75$ m is $\lambda/2$, the sound's wavelength is $\lambda = 1.50$ m. The next point of destructive interference, with $m = 1$, occurs when

$$\Delta\phi = 2\pi\frac{\Delta x}{\lambda} + \Delta\phi_0 = 2\pi\frac{\Delta x}{\lambda} + \pi = 3\pi \text{ rad}$$

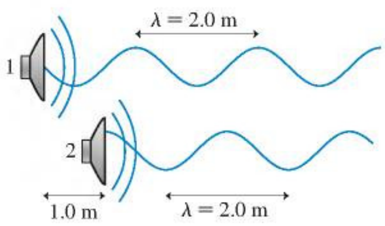
Thus the distance between the speakers when the sound intensity is again a minimum is

$$\Delta x = \lambda = 1.50 \text{ m}$$

ASSESS A separation of λ gives constructive interference for two *identical* speakers ($\Delta\phi_0 = 0$). Here the phase difference of π rad between the speakers (one is pushing forward as the other pulls back) gives destructive interference at this separation.

STOP TO THINK 17.4 Two loudspeakers emit waves with $\lambda = 2.0$ m. Speaker 2 is 1.0 m in front of speaker 1. What, if anything, can be done to cause constructive interference between the two waves?

a. Move speaker 1 forward (to the right) 1.0 m.
b. Move speaker 1 forward (to the right) 0.5 m.
c. Move speaker 1 backward (to the left) 0.5 m.
d. Move speaker 1 backward (to the left) 1.0 m.
e. Nothing. The situation shown already causes constructive interference.
f. Constructive interference is not possible for any placement of the speakers.



17.6 The Mathematics of Interference

Let's look more closely at the superposition of two waves. As two waves of equal amplitude and frequency travel together along the x-axis, the net displacement of the medium is

$$\begin{aligned} D = D_1 + D_2 &= a\sin(kx_1 - \omega t + \phi_{10}) + a\sin(kx_2 - \omega t + \phi_{20}) \\ &= a\sin\phi_1 + a\sin\phi_2 \end{aligned} \qquad (17.24)$$

where the phases ϕ_1 and ϕ_2 were defined in Equations 17.20.

A useful trigonometric identity is

$$\sin\alpha + \sin\beta = 2\cos\left[\tfrac{1}{2}(\alpha - \beta)\right]\sin\left[\tfrac{1}{2}(\alpha + \beta)\right] \qquad (17.25)$$

This identity is certainly not obvious, although it is easily proven by working backward from the right side. We can use this identity to write the net displacement of Equation 17.24 as

$$D = \left[2a\cos\left(\frac{\Delta\phi}{2}\right)\right]\sin(kx_{\text{avg}} - \omega t + (\phi_0)_{\text{avg}}) \qquad (17.26)$$

where $\Delta\phi = \phi_2 - \phi_1$ is the phase difference between the two waves, exactly as in Equation 17.21. $x_{\text{avg}} = (x_1 + x_2)/2$ is the average distance to the two sources and $(\phi_0)_{\text{avg}} = (\phi_{10} + \phi_{20})/2$ is the average phase constant of the sources.

The sine term shows that the superposition of the two waves is still a traveling wave. An observer would see a sinusoidal wave moving along the x-axis with the *same* wavelength and frequency as the original waves.

But how *big* is this wave compared to the two original waves? They each had amplitude a, but the amplitude of their superposition is

$$A = \left|2a\cos\left(\frac{\Delta\phi}{2}\right)\right| \qquad (17.27)$$

where we have used an absolute value sign because amplitudes must be positive. Depending upon the phase difference of the two waves, the amplitude of their superposition can be anywhere from zero (perfect destructive interference) to $2a$ (maximum constructive interference).

The amplitude has its maximum value $A = 2a$ if $\cos(\Delta\phi/2) = \pm 1$. This occurs when

$$\Delta\phi = m \cdot 2\pi \qquad \text{(maximum amplitude } A = 2a\text{)} \qquad (17.28)$$

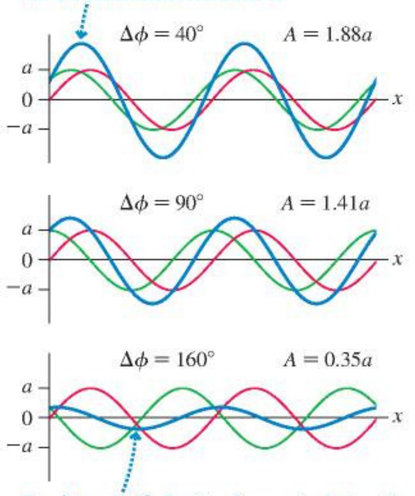
where m is an integer. Similarly, the amplitude is zero if $\cos(\Delta\phi/2) = 0$, which occurs when

$$\Delta\phi = \left(m + \tfrac{1}{2}\right) \cdot 2\pi \qquad \text{(minimum amplitude } A = 0\text{)} \qquad (17.29)$$

Equations 17.28 and 17.29 are identical to the conditions of Equations 17.22 and 17.23 for constructive and destructive interference. We initially found these conditions by

FIGURE 17.22 The interference of two waves for three different values of the phase difference.

For $\Delta\phi = 40°$, the interference is constructive but not maximum constructive.



For $\Delta\phi = 160°$, the interference is destructive but not perfect destructive.

considering the alignment of the crests and troughs. Now we have confirmed them with an algebraic addition of the waves.

It is entirely possible, of course, that the two waves are neither exactly in phase nor exactly out of phase. Equation 17.27 allows us to calculate the amplitude of the superposition for any value of the phase difference. As an example, **FIGURE 17.22** on the previous page shows the calculated interference of two waves that differ in phase by 40°, by 90°, and by 160°.

EXAMPLE 17.8 More interference of sound waves

Two loudspeakers emit 500 Hz sound waves with an amplitude of 0.10 mm. Speaker 2 is 1.00 m behind speaker 1, and the phase difference between the speakers is 90°. What is the amplitude of the sound wave at a point 2.00 m in front of speaker 1?

MODEL The amplitude is determined by the interference of the two waves. Assume that the speed of sound has a room-temperature (20°C) value of 343 m/s.

SOLVE The amplitude of the sound wave is

$$A = |2a\cos(\Delta\phi/2)|$$

where $a = 0.10$ mm and the phase difference between the waves is

$$\Delta\phi = \phi_2 - \phi_1 = 2\pi\frac{\Delta x}{\lambda} + \Delta\phi_0$$

The sound's wavelength is

$$\lambda = \frac{v}{f} = \frac{343\text{ m/s}}{500\text{ Hz}} = 0.686\text{ m}$$

Distances $x_1 = 2.00$ m and $x_2 = 3.00$ m are measured from the speakers, so the path-length difference is $\Delta x = 1.00$ m. We're given that the inherent phase difference between the speakers is $\Delta\phi_0 = \pi/2$ rad. Thus the phase difference at the observation point is

$$\Delta\phi = 2\pi\frac{\Delta x}{\lambda} + \Delta\phi_0 = 2\pi\frac{1.00\text{ m}}{0.686\text{ m}} + \frac{\pi}{2}\text{ rad} = 10.73\text{ rad}$$

and the amplitude of the wave at this point is

$$A = \left|2a\cos\left(\frac{\Delta\phi}{2}\right)\right| = \left|(0.200\text{ mm})\cos\left(\frac{10.73}{2}\right)\right| = 0.121\text{ mm}$$

ASSESS The interference is constructive because $A > a$, but less than maximum constructive interference.

Application: Thin-Film Optical Coatings

The shimmering colors of soap bubbles and oil slicks, as seen in the photo at the beginning of the chapter, are due to the interference of light waves. In fact, the idea of light-wave interference in one dimension has an important application in the optics industry, namely the use of **thin-film optical coatings.** These films, less than 1 μm (10^{-6} m) thick, are placed on glass surfaces, such as lenses, to control reflections from the glass. Antireflection coatings on the lenses in cameras, microscopes, and other optical equipment are examples of thin-film coatings.

FIGURE 17.23 The two reflections, one from the coating and one from the glass, interfere.

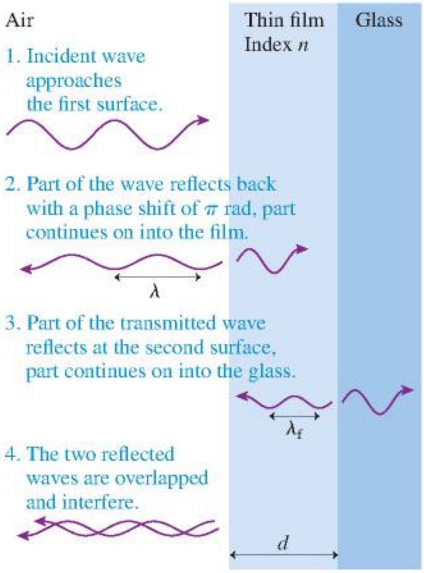


FIGURE 17.23 shows a light wave of wavelength λ approaching a piece of glass that has been coated with a transparent film of thickness d whose index of refraction is n. The air-film boundary is a discontinuity at which the wave speed suddenly decreases, and you saw earlier, in Figure 17.7, that a discontinuity causes a reflection. Most of the light is transmitted into the film, but a little bit is reflected.

Furthermore, you saw in Figure 17.7 that the wave reflected from a discontinuity at which the speed decreases is *inverted* with respect to the incident wave. For a sinusoidal wave, which we're now assuming, the inversion is represented mathematically as a phase shift of π rad. The speed of a light wave decreases when it enters a material with a *larger* index of refraction. Thus **a light wave that reflects from a boundary at which the index of refraction increases has a phase shift of π rad.** There is no phase shift for the reflection from a boundary at which the index of refraction decreases. The reflection in Figure 17.23 is from a boundary between air ($n_{\text{air}} = 1.00$) and a transparent film with $n_{\text{film}} > n_{\text{air}}$, so the reflected wave is inverted due to the phase shift of π rad.

When the transmitted wave reaches the glass, most of it continues on into the glass but a portion is reflected back to the left. We'll assume that the index of refraction of the glass is larger than that of the film, $n_{\text{glass}} > n_{\text{film}}$, so this reflection also has a phase shift of π rad. This second reflection, after traveling back through the film, passes back into the air. There are now *two* equal-frequency reflected waves traveling to the left, and these waves will interfere. If the two reflected waves are *in phase,* they will interfere constructively to cause a *strong reflection.* If the two reflected waves are *out of* *phase,* they will interfere destructively to cause a *weak reflection* or, if their amplitudes are equal, *no reflection* at all.

This suggests practical uses for thin-film optical coatings. The reflections from glass surfaces, even if weak, are often undesirable. For example, reflections degrade the performance of optical equipment. These reflections can be eliminated by coating the glass with a film whose thickness is chosen to cause *destructive* interference of the two reflected waves. This is an *antireflection coating.*

The amplitude of the reflected light depends on the phase difference between the two reflected waves. This phase difference is

$$\begin{aligned}\Delta\phi &= \phi_2 - \phi_1 = (kx_2 + \phi_{20} + \pi \text{ rad}) - (kx_1 + \phi_{10} + \pi \text{ rad}) \\ &= 2\pi\frac{\Delta x}{\lambda_f} + \Delta\phi_0\end{aligned} \tag{17.30}$$

where we explicitly included the reflection phase shift of each wave. In this case, because *both* waves had a phase shift of π rad, the reflection phase shifts cancel.

The wavelength λ_f is the wavelength *in the film* because that's where the path-length difference Δx occurs. You learned in Chapter 16 that the wavelength in a transparent material with index of refraction n is $\lambda_f = \lambda/n$, where the unsubscripted λ is the wavelength in vacuum or air. That is, λ is the wavelength that we measure on "our" side of the air-film boundary.

The path-length difference between the two waves is $\Delta x = 2d$ because wave 2 travels through the film *twice* before rejoining wave 1. The two waves have a common origin—the initial division of the incident wave at the front surface of the film—so the inherent phase difference is $\Delta\phi_0 = 0$. Thus the phase difference of the two reflected waves is

$$\Delta\phi = 2\pi\frac{2d}{\lambda/n} = 2\pi\frac{2nd}{\lambda} \tag{17.31}$$

The interference is constructive, causing a strong reflection, when $\Delta\phi = m\cdot 2\pi$ rad. So when both reflected waves have a phase shift of π rad, constructive interference occurs for wavelengths

$$\lambda_C = \frac{2nd}{m} \qquad m = 1, 2, 3, \ldots \qquad \text{(constructive interference)} \tag{17.32}$$

You will notice that m starts with 1, rather than 0, in order to give meaningful results. Destructive interference, with minimum reflection, requires $\Delta\phi = (m - \frac{1}{2})\cdot 2\pi$ rad. This—again, when both waves have a phase shift of π rad—occurs for wavelengths

$$\lambda_D = \frac{2nd}{m - \frac{1}{2}} \qquad m = 1, 2, 3, \ldots \qquad \text{(destructive interference)} \tag{17.33}$$

We've used $m - \frac{1}{2}$, rather than $m + \frac{1}{2}$, so that m can start with 1 to match the condition for constructive interference.

NOTE The exact condition for constructive or destructive interference is satisfied for only a few discrete wavelengths λ. Nonetheless, reflections are strongly enhanced (nearly constructive interference) for a range of wavelengths near λ_C. Likewise, there is a range of wavelengths near λ_D for which the reflection is nearly canceled.

Antireflection coatings use the interference of light waves to nearly eliminate reflections from glass surfaces.

EXAMPLE 17.9 Designing an antireflection coating

Magnesium fluoride (MgF_2) is used as an antireflection coating on lenses. The index of refraction of MgF_2 is 1.39. What is the thinnest film of MgF_2 that works as an antireflection coating at $\lambda = 510$ nm, near the center of the visible spectrum?

MODEL Reflection is minimized if the two reflected waves interfere destructively.

SOLVE The film thicknesses that cause destructive interference at wavelength λ are

$$d = \left(m - \frac{1}{2}\right)\frac{\lambda}{2n}$$

The thinnest film has $m = 1$. Its thickness is

$$d = \frac{\lambda}{4n} = \frac{510 \text{ nm}}{4(1.39)} = 92 \text{ nm}$$

The film thickness is significantly less than the wavelength of visible light!

Continued

ASSESS The reflected light is completely eliminated (perfect destructive interference) only if the two reflected waves have equal amplitudes. In practice, they don't. Nonetheless, the reflection is reduced from ≈ 4% of the incident intensity for "bare glass" to well under 1%. Furthermore, the intensity of reflected light is much reduced across most of the visible spectrum (400–700 nm), even though the phase difference deviates more and more from π rad as the wavelength moves away from 510 nm. It is the increasing reflection at the ends of the visible spectrum ($\lambda \approx 400$ nm and $\lambda \approx 700$ nm), where $\Delta\phi$ deviates significantly from π rad, that gives a reddish-purple tinge to the lenses on cameras and binoculars. Homework problems will let you explore situations where only one of the two reflections has a reflection phase shift of π rad.

17.7 Interference in Two and Three Dimensions

FIGURE 17.24 A circular or spherical wave.

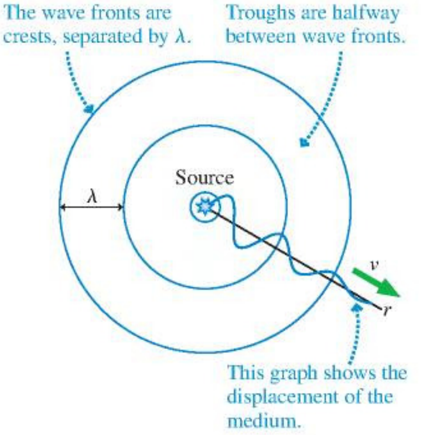
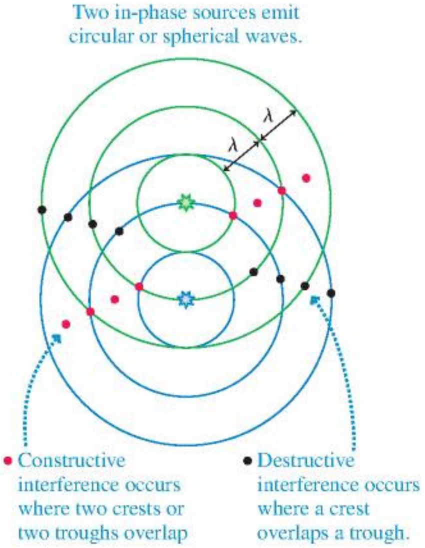


FIGURE 17.25 The overlapping ripple patterns of two sources. Several points of constructive and destructive interference are noted.



Ripples on a lake move in two dimensions. The glow from a lightbulb spreads outward as a spherical wave. A circular or spherical wave, illustrated in **FIGURE 17.24**, can be written

$$D(r,t) = a\sin(kr - \omega t + \phi_0) \tag{17.34}$$

where r is the distance measured outward from the source. Equation 17.34 is our familiar wave equation with the one-dimensional coordinate x replaced by a more general radial coordinate r. Recall that the wave fronts represent the *crests* of the wave and are spaced by the wavelength λ.

What happens when two circular or spherical waves overlap? For example, imagine two paddles oscillating up and down on the surface of a pond. We will assume that the two paddles oscillate with the same frequency and amplitude and that they are in phase. **FIGURE 17.25** shows the wave fronts of the two waves. The ripples overlap as they travel, and, as was the case in one dimension, this causes interference. An important difference, though, is that amplitude decreases with distance as waves spread out in two or three dimensions—a consequence of energy conservation—so the two overlapped waves generally do *not* have equal amplitudes. Consequently, destructive interference rarely produces perfect cancellation.

Maximum constructive interference occurs where two crests align or two troughs align. Several locations of constructive interference are marked in Figure 17.25. Intersecting wave fronts are points where two crests are aligned. It's a bit harder to visualize, but two troughs are aligned when a midpoint between two wave fronts is overlapped with another midpoint between two wave fronts. Maximum, but usually not perfect, destructive interference occurs where the crest of one wave aligns with a trough of the other wave. Several points of destructive interference are also indicated in Figure 17.25.

A picture on a page is static, but **the wave fronts are in motion**. Try to imagine the wave fronts of Figure 17.25 expanding outward as new circular rings are born at the sources. The waves will move forward half a wavelength during half a period, causing the crests in Figure 17.25 to be replaced by troughs while the troughs become crests.

The important point to recognize is that **the motion of the waves does not affect the points of constructive and destructive interference.** Points in the figure where two crests overlap will become points where two troughs overlap, but this overlap is still constructive interference. Similarly, points in the figure where a crest and a trough overlap will become a point where a trough and a crest overlap—still destructive interference.

The mathematical description of interference in two or three dimensions is very similar to that of one-dimensional interference. The net displacement of a particle in the medium is

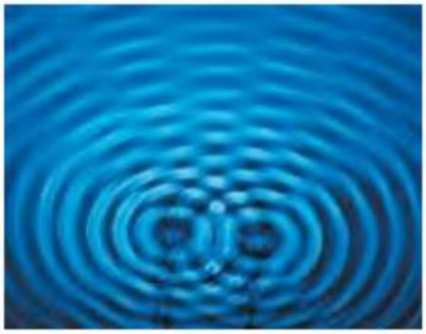
$$D = D_1 + D_2 = a_1\sin(kr_1 - \omega t + \phi_{10}) + a_2\sin(kr_2 - \omega t + \phi_{20}) \tag{17.35}$$

The only differences between Equation 17.35 and the earlier one-dimensional Equation 17.24 are that the linear coordinates have been changed to radial coordinates and we've allowed the amplitudes to differ. These changes do not affect the phase difference, which, with x replaced by r, is now

$$\Delta\phi = 2\pi \frac{\Delta r}{\lambda} + \Delta\phi_0 \tag{17.36}$$

The term $2\pi(\Delta r/\lambda)$ is the phase difference that arises when the waves travel different distances from the sources to the point at which they combine. Δr itself is the *path-length difference.* As before, $\Delta\phi_0$ is any inherent phase difference of the sources themselves.

Maximum constructive interference occurs, just as in one dimension, at those points where $\cos(\Delta\phi/2) = \pm 1$. Similarly, maximum destructive interference occurs at points where $\cos(\Delta\phi/2) = 0$. The conditions for constructive and destructive interference are



Two overlapping water waves create an interference pattern.

Maximum constructive interference:

$$\Delta\phi = 2\pi \frac{\Delta r}{\lambda} + \Delta\phi_0 = m \cdot 2\pi$$

$$m = 0, 1, 2, \ldots \tag{17.37}$$

Maximum destructive interference:

$$\Delta\phi = 2\pi \frac{\Delta r}{\lambda} + \Delta\phi_0 = \left(m + \tfrac{1}{2}\right) \cdot 2\pi$$

Identical Sources

For two identical sources (i.e., sources that oscillate in phase with $\Delta\phi_0 = 0$), the conditions for constructive and destructive interference are simple:

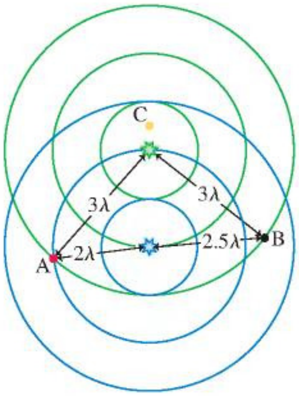
$$\begin{aligned} &\text{Constructive:} \quad \Delta r = m\lambda \\ &\text{Destructive:} \quad \Delta r = \left(m + \frac{1}{2}\right)\lambda \end{aligned} \qquad \text{(identical sources)} \tag{17.38}$$

The waves from two identical sources interfere constructively at points where the path-length difference is an integer number of wavelengths because, for these values of Δr, crests are aligned with crests and troughs with troughs. **The waves interfere destructively at points where the path-length difference is a half-integer number of wavelengths** because, for these values of Δr, crests are aligned with troughs. These two statements are the essence of interference.

NOTE Equation 17.38 applies only if the sources are in phase. If the sources are not in phase, you must use the more general Equation 17.37 to locate the points of constructive and destructive interference.

FIGURE 17.26 For identical sources, the path-length difference Δr determines whether the interference at a particular point is constructive or destructive.

- At A, $\Delta r_A = \lambda$, so this is a point of constructive interference.



- At B, $\Delta r_B = \frac{1}{2}\lambda$, so this is a point of destructive interference.

Wave fronts are spaced exactly one wavelength apart; hence we can measure the distances r_1 and r_2 simply by counting the rings in the wave-front pattern. In **FIGURE 17.26**, which is based on Figure 17.25, point A is distance $r_1 = 3\lambda$ from the first source and $r_2 = 2\lambda$ from the second. The path-length difference is $\Delta r_A = 1\lambda$, the condition for the maximum constructive interference of identical sources. Point B has $\Delta r_B = \frac{1}{2}\lambda$, so it is a point of maximum destructive interference.

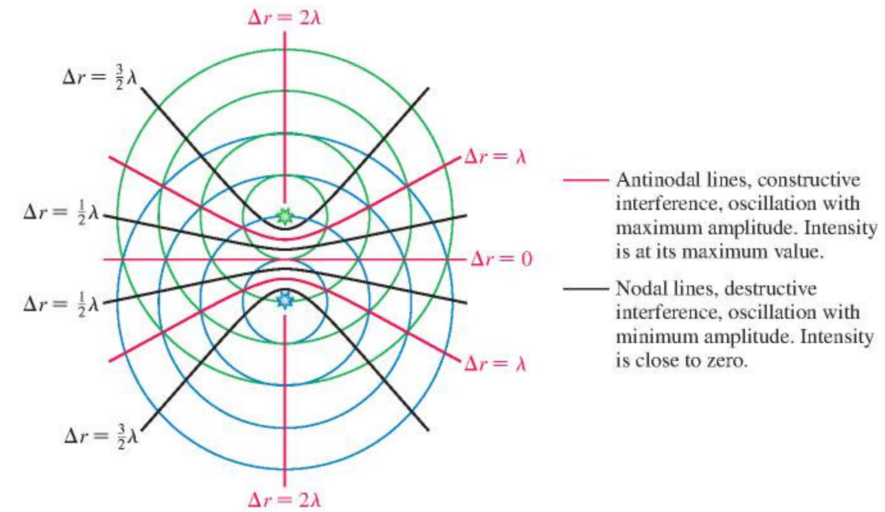
NOTE Interference is determined by Δr, the path-length *difference,* rather than by r_1 or r_2.

STOP TO THINK 17.5 The interference at point C in Figure 17.26 is

a. Maximum constructive.
b. Constructive, but less than maximum.
c. Maximum destructive.
d. Destructive, but less than maximum.
e. There is no interference at point C.

We can now locate the points of maximum constructive interference by drawing a line through *all* the points at which $\Delta r = 0$, another line through all the points at which $\Delta r = \lambda$, and so on. These lines, shown in red in FIGURE 17.27, are called **antinodal lines.** They are analogous to the antinodes of a standing wave, hence the name. An antinode is a *point* of maximum constructive interference; for circular waves, oscillation at maximum amplitude occurs along a continuous *line*. Similarly, destructive interference occurs along lines called **nodal lines.** The amplitude is a minimum along a nodal line, usually close to zero, just as it is at a node in a standing-wave pattern.

FIGURE 17.27 The points of constructive and destructive interference fall along antinodal and nodal lines.



Antinodal lines, constructive interference, oscillation with maximum amplitude. Intensity is at its maximum value.

Nodal lines, destructive interference, oscillation with minimum amplitude. Intensity is close to zero.

A Problem-Solving Strategy for Interference Problems

The information in this section is the basis of a strategy for solving interference problems. This strategy applies equally well to interference in one dimension if you use Δx instead of Δr.

PROBLEM-SOLVING STRATEGY 17.1

Interference of two waves

MODEL Model the waves as linear, circular, or spherical.

VISUALIZE Draw a picture showing the sources of the waves and the point where the waves interfere. Give relevant dimensions. Identify the distances r_1 and r_2 from the sources to the point. Note any phase difference $\Delta\phi_0$ between the two sources.

SOLVE The interference depends on the path-length difference $\Delta r = r_2 - r_1$ and the source phase difference $\Delta\phi_0$.

$$\text{Constructive:} \quad \Delta\phi = 2\pi\frac{\Delta r}{\lambda} + \Delta\phi_0 = m \cdot 2\pi$$

$$m = 0, 1, 2, \ldots$$

$$\text{Destructive:} \quad \Delta\phi = 2\pi\frac{\Delta r}{\lambda} + \Delta\phi_0 = \left(m + \tfrac{1}{2}\right) \cdot 2\pi$$

For identical sources ($\Delta\phi_0 = 0$), the interference is maximum constructive if $\Delta r = m\lambda$, maximum destructive if $\Delta r = (m + \frac{1}{2})\lambda$.

ASSESS Check that your result has correct units and significant figures, is reasonable, and answers the question.

Exercise 18

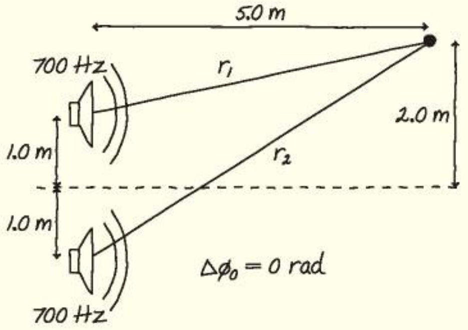
EXAMPLE 17.10 Two-dimensional interference between two loudspeakers

Two loudspeakers in a plane are 2.0 m apart and in phase with each other. Both emit 700 Hz sound waves into a room where the speed of sound is 341 m/s. A listener stands 5.0 m in front of the loudspeakers and 2.0 m to one side of the center. Is the interference at this point maximum constructive, maximum destructive, or in between? How will the situation differ if the loudspeakers are out of phase?

MODEL The two speakers are sources of in-phase, spherical waves. The overlap of these waves causes interference.

VISUALIZE FIGURE 17.28 shows the loudspeakers and defines the distances r_1 and r_2 to the point of observation. The figure includes dimensions and notes that $\Delta\phi_0 = 0$ rad.

FIGURE 17.28 Pictorial representation of the interference between two loudspeakers.



SOLVE It's not r_1 and r_2 that matter, but the *difference* Δr between them. From the geometry of the figure we can calculate that

$$r_1 = \sqrt{(5.0\text{ m})^2 + (1.0\text{ m})^2} = 5.10\text{ m}$$
$$r_2 = \sqrt{(5.0\text{ m})^2 + (3.0\text{ m})^2} = 5.83\text{ m}$$

Thus the path-length difference is $\Delta r = r_2 - r_1 = 0.73$ m. The wavelength of the sound waves is

$$\lambda = \frac{v}{f} = \frac{341\text{ m/s}}{700\text{ Hz}} = 0.487\text{ m}$$

In terms of wavelengths, the path-length difference is $\Delta r/\lambda = 1.50$, or

$$\Delta r = \tfrac{3}{2}\lambda$$

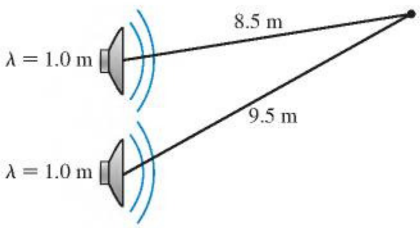
Because the sources are in phase ($\Delta\phi_0 = 0$), this is the condition for *destructive* interference. If the sources were out of phase ($\Delta\phi_0 = \pi$ rad), then the phase difference of the waves at the listener would be

$$\Delta\phi = 2\pi\frac{\Delta r}{\lambda} + \Delta\phi_0 = 2\pi\left(\frac{3}{2}\right) + \pi\text{ rad} = 4\pi\text{ rad}$$

This is an integer multiple of 2π rad, so in this case the interference would be *constructive.*

ASSESS Both the path-length difference and any inherent phase difference of the sources must be considered when evaluating interference.

STOP TO THINK 17.6 These two loudspeakers are in phase. They emit equal-amplitude sound waves with a wavelength of 1.0 m. At the point indicated, is the interference maximum constructive, maximum destructive, or something in between?



17.8 Beats

Thus far we have looked at the superposition of sources having the same wavelength and frequency. We can also use the principle of superposition to investigate a phenomenon that is easily demonstrated with two sources of slightly different frequency.

If you listen to two sounds with very different frequencies, such as a high note and a low note, you hear two distinct tones. But if the frequency difference is very small, just one or two hertz, then you hear a single tone whose intensity is *modulated* once or twice every second. That is, the sound goes up and down in volume, loud, soft, loud, soft, … , making a distinctive sound pattern called **beats.**

Consider two sinusoidal waves traveling along the x-axis with angular frequencies $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$. The two waves are

$$\begin{aligned} D_1 &= a\sin(k_1x - \omega_1t + \phi_{10}) \\ D_2 &= a\sin(k_2x - \omega_2t + \phi_{20}) \end{aligned} \qquad (17.39)$$

where the subscripts 1 and 2 indicate that the frequencies, wave numbers, and phase constants of the two waves may be different.

To simplify the analysis, let's make several assumptions:

1. The two waves have the same amplitude a,
2. A detector, such as your ear, is located at the origin ($x = 0$),
3. The two sources are in phase ($\phi_{10} = \phi_{20}$), and
4. The source phases happen to be $\phi_{10} = \phi_{20} = \pi$ rad.

None of these assumptions is essential to the outcome. All could be otherwise and we would still come to basically the same conclusion, but the mathematics would be far messier. Making these assumptions allows us to emphasize the physics with the least amount of mathematics.

With these assumptions, the two waves as they reach the detector at $x = 0$ are

$$\begin{aligned} D_1 &= a\sin(-\omega_1 t + \pi) = a\sin\omega_1 t \\ D_2 &= a\sin(-\omega_2 t + \pi) = a\sin\omega_2 t \end{aligned} \qquad (17.40)$$

where we've used the trigonometric identity $\sin(\pi - \theta) = \sin\theta$. The principle of superposition tells us that the *net* displacement of the medium at the detector is the sum of the displacements of the individual waves. Thus

$$D = D_1 + D_2 = a(\sin\omega_1 t + \sin\omega_2 t) \qquad (17.41)$$

Earlier, for interference, we used the trigonometric identity

$$\sin\alpha + \sin\beta = 2\cos\left[\tfrac{1}{2}(\alpha - \beta)\right]\sin\left[\tfrac{1}{2}(\alpha + \beta)\right]$$

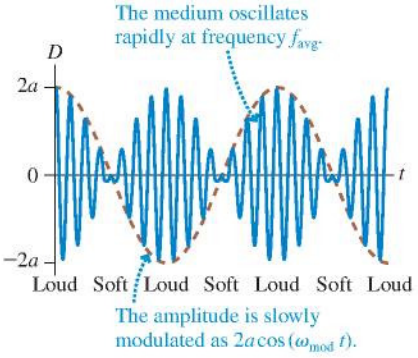
We can use this identity again to write Equation 17.41 as

$$\begin{aligned} D &= 2a\cos\left[\tfrac{1}{2}(\omega_1 - \omega_2)t\right]\sin\left[\tfrac{1}{2}(\omega_1 + \omega_2)t\right] \\ &= \left[2a\cos(\omega_{\text{mod}} t)\right]\sin(\omega_{\text{avg}} t) \end{aligned} \qquad (17.42)$$

where $\omega_{\text{avg}} = \frac{1}{2}(\omega_1 + \omega_2)$ is the *average* angular frequency and $\omega_{\text{mod}} = \frac{1}{2}|\omega_1 - \omega_2|$ is called the *modulation frequency*. We've used the absolute value because the modulation depends only on the frequency *difference* between the sources, not on which has the larger frequency.

We are interested in the situation when the two frequencies are very nearly equal: $\omega_1 \approx \omega_2$. In that case, ω_{avg} hardly differs from either ω_1 or ω_2 while ω_{mod} is very near to—but not exactly—zero. When ω_{mod} is very small, the term $\cos(\omega_{\text{mod}} t)$ oscillates *very* slowly. We have grouped it with the $2a$ term because, together, they provide a slowly changing "amplitude" for the rapid oscillation at frequency ω_{avg}.

FIGURE 17.29 Beats are caused by the superposition of two waves of nearly identical frequency.



The medium oscillates rapidly at frequency f_{avg}.

The amplitude is slowly modulated as $2a\cos(\omega_{\text{mod}} t)$.

FIGURE 17.29 is a history graph of the wave at the detector ($x = 0$). It shows the oscillation of the air against your eardrum at frequency $f_{\text{avg}} = \omega_{\text{avg}}/2\pi = \frac{1}{2}(f_1 + f_2)$. This oscillation determines the note you hear; it differs little from the two notes at frequencies f_1 and f_2. We are especially interested in the time-dependent amplitude, shown as a dashed line, that is given by the term $2a\cos(\omega_{\text{mod}} t)$. This periodically varying amplitude is called a **modulation** of the wave, which is where ω_{mod} gets its name.

As the amplitude rises and falls, the sound alternates as loud, soft, loud, soft, and so on. But that is exactly what you hear when you listen to beats! The alternating loud and soft sounds arise from the two waves being alternately in phase and out of phase, causing constructive and then destructive interference.

Imagine two people walking side by side at just slightly different paces. Initially both of their right feet hit the ground together, but after a while they get out of step. A little bit later they are back in step and the process alternates. The sound waves are doing the same. Initially the crests of each wave, of amplitude a, arrive together at your ear and the net displacement is doubled to $2a$. But after a while the two waves, being of slightly different frequency, get out of step and a crest of one arrives with a trough of the other. When this happens, the two waves cancel each other to give a net displacement of zero. This process alternates over and over, loud and soft.

Notice, in Figure 17.29, that the sound intensity rises and falls *twice* during one cycle of the modulation envelope. Each "loud-soft-loud" is one beat, so the **beat frequency** f_{beat}, which is the number of beats per second, is *twice* the modulation frequency $f_{\text{mod}} = \omega_{\text{mod}}/2\pi$. From the above definition of ω_{mod}, the beat frequency is

$$f_{\text{beat}} = 2f_{\text{mod}} = 2\frac{\omega_{\text{mod}}}{2\pi} = 2\cdot\frac{1}{2}\left(\frac{\omega_1}{2\pi} - \frac{\omega_2}{2\pi}\right) = |f_1 - f_2| \qquad (17.43)$$

The beat frequency is simply the *difference* between the two individual frequencies.

EXAMPLE 17.11 | **Detecting bats with beats**

The little brown bat is a common species in North America. It emits echolocation pulses at a frequency of 40 kHz, well above the range of human hearing. To allow researchers to "hear" these bats, the bat detector shown in **FIGURE 17.30** combines the bat's sound wave at frequency f_1 with a wave of frequency f_2 from a tunable oscillator. The resulting beat frequency is then amplified and sent to a loudspeaker. To what frequency should the tunable oscillator be set to produce an audible beat frequency of 3 kHz?

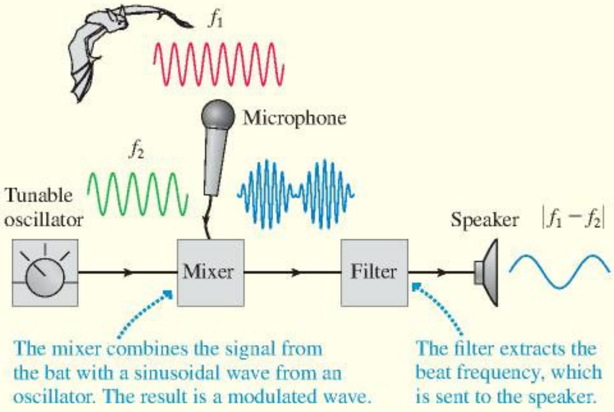
SOLVE Combining two waves with different frequencies gives a beat frequency

$$f_{\text{beat}} = |f_1 - f_2|$$

A beat frequency will be generated at 3 kHz if the oscillator frequency and the bat frequency *differ* by 3 kHz. An oscillator frequency of either 37 kHz or 43 kHz will work nicely.

ASSESS The electronic circuitry of radios, televisions, and cell phones makes extensive use of *mixers* to generate difference frequencies.

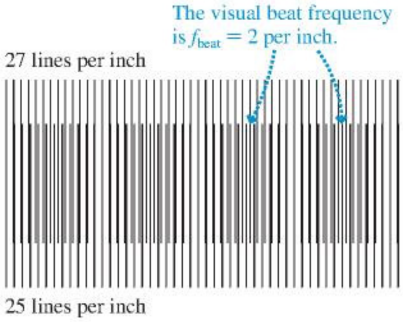
FIGURE 17.30 The operation of a bat detector.



Beats aren't limited to sound waves. **FIGURE 17.31** shows a graphical example of beats. Two "fences" of slightly different frequencies are superimposed on each other. The difference in the two frequencies is two lines per inch. You can confirm, with a ruler, that the figure has two "beats" per inch, in agreement with Equation 17.43.

Beats are important in many other situations. For example, you have probably seen movies where rotating wheels seem to turn slowly backward. Why is this? Suppose the movie camera is shooting at 30 frames per second but the wheel is rotating 32 times per second. The combination of the two produces a "beat" of 2 Hz, meaning that the wheel appears to rotate only twice per second. The same is true if the wheel is rotating 28 times per second, but in this case, where the wheel frequency slightly lags the camera frequency, it appears to rotate *backward* twice per second!

FIGURE 17.31 A graphical example of beats.



STOP TO THINK 17.7 You hear three beats per second when two sound tones are generated. The frequency of one tone is 610 Hz. The frequency of the other is

a. 604 Hz
b. 607 Hz
c. 613 Hz
d. 616 Hz
e. Either a or d.
f. Either b or c.

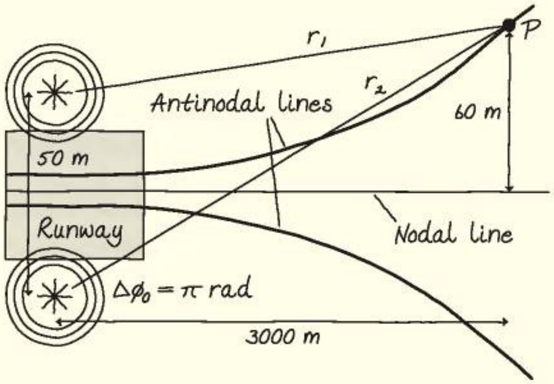
CHALLENGE EXAMPLE 17.12 An airplane landing system

Your firm has been hired to design a system that allows airplane pilots to make instrument landings in rain or fog. You've decided to place two radio transmitters 50 m apart on either side of the runway. These two transmitters will broadcast the same frequency, but out of phase with each other. This will cause a nodal line to extend straight off the end of the runway. As long as the airplane's receiver is silent, the pilot knows she's directly in line with the runway. If she drifts to one side or the other, the radio will pick up a signal and sound a warning beep. To have sufficient accuracy, the first intensity maxima need to be 60 m on either side of the nodal line at a distance of 3.0 km. What frequency should you specify for the transmitters?

MODEL The two transmitters are sources of out-of-phase, circular waves. The overlap of these waves produces an interference pattern.

VISUALIZE For out-of-phase sources, the center line—with zero path-length difference—is a nodal line of maximum destructive interference because the two signals always arrive out of phase. FIGURE 17.32 shows the nodal line, extending straight off the runway, and the first antinodal line—the points of maximum constructive interference—on either side. Comparing this to Figure 17.27, where the two sources were in phase, you can see that the nodal and antinodal lines have been reversed.

FIGURE 17.32 Pictorial representation of the landing system.



SOLVE Point P, 60 m to the side at a distance of 3000 m, needs to be a point of maximum constructive interference. The distances are

$$r_1 = \sqrt{(3000\text{ m})^2 + (60\text{ m} - 25\text{ m})^2} = 3000.204\text{ m}$$
$$r_2 = \sqrt{(3000\text{ m})^2 + (60\text{ m} + 25\text{ m})^2} = 3001.204\text{ m}$$

We needed to keep several extra significant figures because we're looking for the difference between two numbers that are almost the same. The path-length difference at P is

$$\Delta r = r_2 - r_1 = 1.000\text{ m}$$

We know, for out-of-phase transmitters, that the phase difference of the sources is $\Delta\phi_0 = \pi$ rad. The first maximum will occur where the phase difference between the waves is $\Delta\phi = 1 \cdot 2\pi$ rad. Thus the condition that we must satisfy at P is

$$\Delta\phi = 2\pi\text{ rad} = 2\pi\frac{\Delta r}{\lambda} + \pi\text{ rad}$$

Solving for λ, we find

$$\lambda = 2\,\Delta r = 2.00\text{ m}$$

Consequently, the required frequency is

$$f = \frac{c}{\lambda} = \frac{3.00 \times 10^8\text{ m/s}}{2.00\text{ m}} = 1.50 \times 10^8\text{ Hz} = 150\text{ MHz}$$

ASSESS 150 MHz is slightly higher than the frequencies of FM radio ($\approx$ 100 MHz) but is well within the radio frequency range. Notice that the condition to be satisfied at P is that the path-length difference must be $\frac{1}{2}\lambda$. This makes sense. A path-length difference of $\frac{1}{2}\lambda$ contributes π rad to the phase difference. When combined with the π rad from the out-of-phase sources, the total phase difference of 2π rad creates constructive interference.

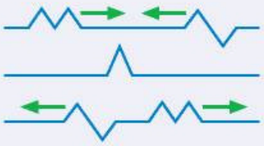
SUMMARY

The goal of Chapter 17 has been to understand and use the idea of superposition.

GENERAL PRINCIPLES

Principle of Superposition

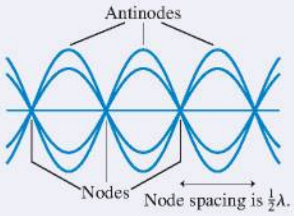
The displacement of a medium when more than one wave is present is the sum at each point of the displacements due to each individual wave.



IMPORTANT CONCEPTS

Standing Waves

Standing waves are due to the superposition of two traveling waves moving in opposite directions.

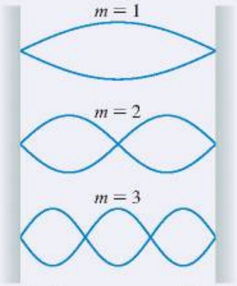


The amplitude at position x is

$$A(x) = 2a\sin kx$$

where a is the amplitude of each wave.

The boundary conditions determine which standing-wave frequencies and wavelengths are allowed. The allowed standing waves are **modes** of the system.



Standing waves on a string

Solving Interference Problems

Maximum constructive interference occurs where crests are aligned with crests and troughs with troughs. The waves are in phase.

Maximum destructive interference occurs where crests are aligned with troughs. The waves are out of phase.

MODEL Model the wave as linear, circular, or spherical.

VISUALIZE Find distances to the sources.

SOLVE Interference depends on the **phase difference** $\Delta\phi$ between the waves:

$$\text{Constructive: } \Delta\phi = 2\pi\frac{\Delta r}{\lambda} + \Delta\phi_0 = m \cdot 2\pi$$

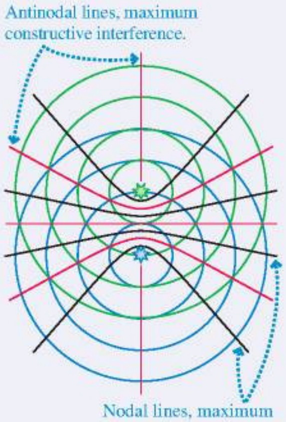
$$\text{Destructive: } \Delta\phi = 2\pi\frac{\Delta r}{\lambda} + \Delta\phi_0 = \left(m + \tfrac{1}{2}\right) \cdot 2\pi$$

Δr is the path-length difference of the two waves, and $\Delta\phi_0$ is any phase difference between the sources. For identical (in-phase) sources:

$$\text{Constructive: } \Delta r = m\lambda \qquad \text{Destructive: } \Delta r = \left(m + \tfrac{1}{2}\right)\lambda$$

ASSESS Is the result reasonable?

Antinodal lines, maximum constructive interference.



Nodal lines, maximum destructive interference.

APPLICATIONS

Boundary conditions

Strings, electromagnetic waves, and sound waves in closed-closed tubes must have nodes at both ends:

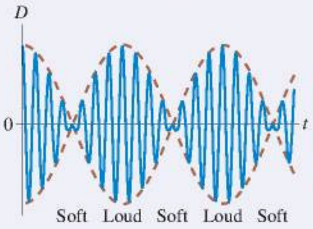
$$\lambda_m = \frac{2L}{m} \qquad f_m = m\frac{v}{2L} = mf_1 \qquad m = 1, 2, 3, \ldots$$

The frequencies and wavelengths are the same for a sound wave in an open-open tube, which has antinodes at both ends.

A sound wave in an open-closed tube must have a node at the closed end but an antinode at the open end. This leads to

$$\lambda_m = \frac{4L}{m} \qquad f_m = m\frac{v}{4L} = mf_1 \qquad m = 1, 3, 5, 7, \ldots$$

Beats (loud-soft-loud-soft modulations of intensity) occur when two waves of slightly different frequency are superimposed.



The beat frequency between waves of frequencies f_1 and f_2 is

$$f_{\text{beat}} = |f_1 - f_2|$$

TERMS AND NOTATION

principle of superposition
standing wave
node
antinode
amplitude function, $A(x)$
boundary condition
fundamental frequency, f_1
harmonic
mode
interference
in phase
constructive interference
out of phase
destructive interference
phase difference, $\Delta\phi$
path-length difference, Δx or Δr
thin-film optical coating
antinodal line
nodal line
beats
modulation
beat frequency, f_{beat}

CONCEPTUAL QUESTIONS

1. **FIGURE Q17.1** shows a standing wave oscillating on a string at frequency f_0.
   a. What mode (m-value) is this?
   b. How many antinodes will there be if the frequency is doubled to $2f_0$?

FIGURE Q17.1

2. If you take snapshots of a standing wave on a string, there are certain instants when the string is totally flat. What has happened to the energy of the wave at those instants?
3. **FIGURE Q17.3** shows the displacement of a standing sound wave in a 32-cm-long horizontal tube of air open at both ends.
   a. What mode (m-value) is this?
   b. Are the air molecules moving horizontally or vertically? Explain.
   c. At what distances from the left end of the tube do the molecules oscillate with maximum amplitude?
   d. At what distances from the left end of the tube does the air pressure oscillate with maximum amplitude?

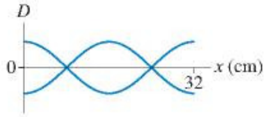


FIGURE Q17.3

4. An organ pipe is tuned to exactly 384 Hz when the room temperature is 20°C. If the room temperature later increases to 22°C, does the pipe's frequency increase, decrease, or stay the same? Explain.
5. If you pour liquid into a tall, narrow glass, you may hear sound with a steadily rising pitch. What is the source of the sound? And why does the pitch rise as the glass fills?
6. A flute filled with helium will, until the helium escapes, play notes at a much higher pitch than normal. Why?
7. In music, two notes are said to be an *octave* apart when one note is exactly twice the frequency of the other. Suppose you have a guitar string playing frequency f_0. To increase the frequency by an octave, to $2f_0$, by what factor would you have to (a) increase the tension or (b) decrease the length?
8. **FIGURE Q17.8** is a snapshot graph of two plane waves passing through a region of space. Each wave has a 2.0 mm amplitude and the same wavelength. What is the net displacement of the medium at points a, b, and c?

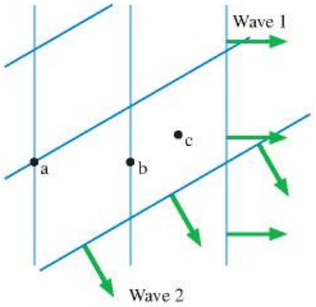


FIGURE Q17.8

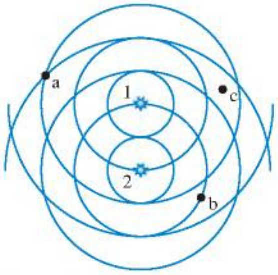


FIGURE Q17.9

9. **FIGURE Q17.9** shows the circular waves emitted by two in-phase sources. Are a, b, and c points of maximum constructive interference, maximum destructive interference, or in between?
10. A trumpet player hears 5 beats per second when she plays a note and simultaneously sounds a 440 Hz tuning fork. After pulling her tuning valve out to slightly increase the length of her trumpet, she hears 3 beats per second against the tuning fork. Was her initial frequency 435 Hz or 445 Hz? Explain.

EXERCISES AND PROBLEMS

Problems labeled ▨ integrate material from earlier chapters.

Exercises

Section 17.1 The Principle of Superposition

1. | **FIGURE EX17.1** is a snapshot graph at $t = 0$ s of two waves approaching each other at 1.0 m/s. Draw six snapshot graphs, stacked vertically, showing the string at 1 s intervals from $t = 1$ s to $t = 6$ s.

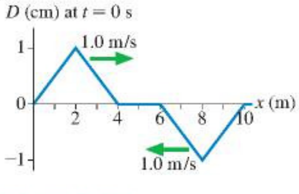


FIGURE EX17.1

2. | **FIGURE EX17.2** is a snapshot graph at $t = 0$ s of two waves approaching each other at 1.0 m/s. Draw six snapshot graphs, stacked vertically, showing the string at 1 s intervals from $t = 1$ s to $t = 6$ s.

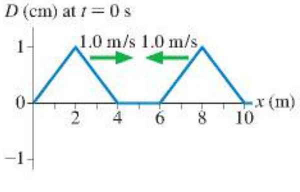


FIGURE EX17.2

3. || **FIGURE EX17.3a** is a snapshot graph at $t = 0$ s of two waves approaching each other at 1.0 m/s. At what time was the snapshot graph in **FIGURE EX17.3b** taken?

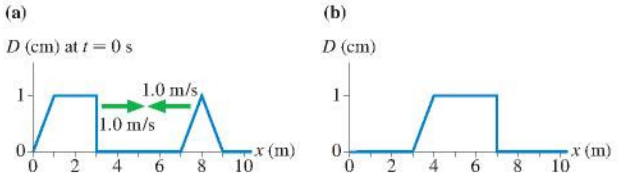


FIGURE EX17.3

Section 17.2 Standing Waves

Section 17.3 Standing Waves on a String

4. || **FIGURE EX17.4** is a snapshot graph at $t = 0$ s of two waves moving to the right at 1.0 m/s. The string is fixed at $x = 8.0$ m. Draw four snapshot graphs, stacked vertically, showing the string at $t = 2$, 4, 6, and 8 s.

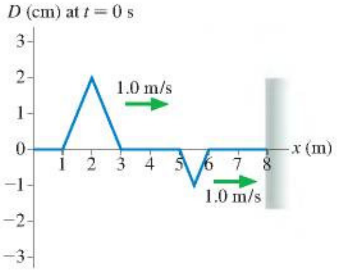


FIGURE EX17.4

5. || **FIGURE EX17.5** shows a standing wave oscillating at 100 Hz on a string. What is the wave speed?

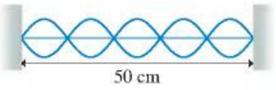


FIGURE EX17.5

FIGURE EX17.6

6. || **FIGURE EX17.6** shows a standing wave on a 2.0-m-long string that has been fixed at both ends and tightened until the wave speed is 40 m/s. What is the frequency?

7. | **FIGURE EX17.7** shows a standing wave on a string that is oscillating at 100 Hz.

FIGURE EX17.7

   a. How many antinodes will there be if the frequency is increased to 200 Hz?
   b. If the tension is increased by a factor of 4, at what frequency will the string continue to oscillate as a standing wave that looks like the one in the figure?

8. | a. What are the three longest wavelengths for standing waves on a 240-cm-long string that is fixed at both ends?
   b. If the frequency of the second-longest wavelength is 50 Hz, what is the frequency of the third-longest wavelength?

9. | Standing waves on a 1.0-m-long string that is fixed at both ends are seen at successive frequencies of 36 Hz and 48 Hz.
   a. What are the fundamental frequency and the wave speed?
   b. Draw the standing-wave pattern when the string oscillates at 48 Hz.

10. | The two highest-pitch strings on a violin are tuned to 440 Hz (the A string) and 659 Hz (the E string). What is the ratio of the mass of the A string to that of the E string? Violin strings are all the same length and under essentially the same tension.

11. || A heavy piece of hanging sculpture is suspended by a 90-cm-long, 5.0 g steel wire. When the wind blows hard, the wire hums at its fundamental frequency of 80 Hz. What is the mass of the sculpture?

12. | A carbon dioxide laser is an infrared laser. A CO_2 laser with a cavity length of 53.00 cm oscillates in the $m = 100{,}000$ mode. What are the wavelength and frequency of the laser beam?

13. | Microwaves pass through a small hole into the "microwave cavity" of **FIGURE EX17.13**. What frequencies between 10 GHz and 20 GHz will create standing waves in the cavity?

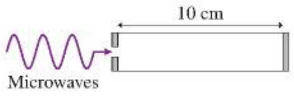


FIGURE EX17.13

Section 17.4 Standing Sound Waves and Musical Acoustics

14. | What are the three longest wavelengths for standing sound waves in a 121-cm-long tube that is (a) open at both ends and (b) open at one end, closed at the other?

15. | **FIGURE EX17.15** shows a standing sound wave in an 80-cm-long tube. The tube is filled with an unknown gas. What is the speed of sound in this gas?

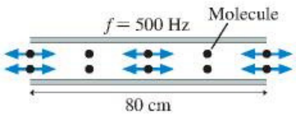


FIGURE EX17.15

16. | The fundamental frequency of an open-open tube is 1500 Hz when the tube is filled with 0°C helium. What is its frequency when filled with 0°C air?

17. | We can make a simple model of the human vocal tract as an open-closed tube extending from the opening of the mouth to the diaphragm. What is the length of this tube if its fundamental frequency equals a typical speech frequency of 250 Hz? The speed of sound in the warm air is 350 m/s.
BIO

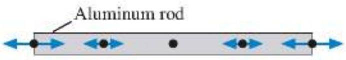
18. || The lowest note on a grand piano has a frequency of 27.5 Hz. The entire string is 2.00 m long and has a mass of 400 g. The vibrating section of the string is 1.90 m long. What tension is needed to tune this string properly?

19. | A bass clarinet can be modeled as a 120-cm-long open-closed tube. A bass clarinet player starts playing in a 20°C room, but soon the air inside the clarinet warms to where the speed of sound is 352 m/s. Does the fundamental frequency increase or decrease? By how much?

20. || A violin string is 30 cm long. It sounds the musical note A (440 Hz) when played without fingering. How far from the end of the string should you place your finger to play the note C (523 Hz)?

21. | A longitudinal standing wave can be created in a long, thin aluminum rod by stroking the rod with very dry fingers. This is often done as a physics demonstration, creating a high-pitched, very annoying whine. From a wave perspective, the standing wave is equivalent to a sound standing wave in an open-open tube. As **FIGURE EX17.21** shows, both ends of the rod are antinodes. What is the fundamental frequency of a 2.0-m-long aluminum rod?

FIGURE EX17.21



Section 17.5 Interference in One Dimension

Section 17.6 The Mathematics of Interference

22. | Two loudspeakers emit sound waves along the x-axis. The sound has maximum intensity when the speakers are 20 cm apart. The sound intensity decreases as the distance between the speakers is increased, reaching zero at a separation of 60 cm.
    a. What is the wavelength of the sound?
    b. If the distance between the speakers continues to increase, at what separation will the sound intensity again be a maximum?
23. || Two loudspeakers in a 20°C room emit 686 Hz sound waves along the x-axis.
    a. If the speakers are in phase, what is the smallest distance between the speakers for which the interference of the sound waves is maximum destructive?
    b. If the speakers are out of phase, what is the smallest distance between the speakers for which the interference of the sound waves is maximum constructive?
24. | Noise-canceling headphones are an application of destructive interference. Each side of the headphones uses a microphone to pick up noise, delays it slightly, then rebroadcasts the noise next to your ear where it can interfere with the incoming sound wave of the noise. Suppose you are sitting 1.8 m from an annoying, 110 Hz buzzing sound. What is the minimum headphone delay, in ms, that will cancel this noise?
25. | What is the thinnest film of MgF_2 $(n = 1.39)$ on glass that produces a strong reflection for orange light with a wavelength of 600 nm?
26. || A very thin oil film $(n = 1.25)$ floats on water $(n = 1.33)$. What is the thinnest film that produces a strong reflection for green light with a wavelength of 500 nm?

Section 17.7 Interference in Two and Three Dimensions

27. || **FIGURE EX17.27** shows the circular wave fronts emitted by two wave sources.
    a. Are these sources in phase or out of phase? Explain.
    b. Make a table with rows labeled P, Q, and R and columns labeled r_1, r_2, Δr, and C/D. Fill in the table for points P, Q, and R, giving the distances as multiples of λ and indicating, with a C or a D, whether the interference at that point is constructive or destructive.

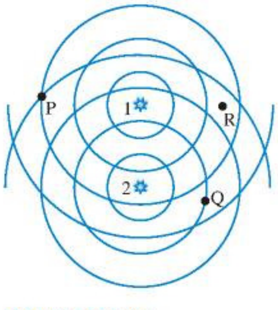


FIGURE EX17.27

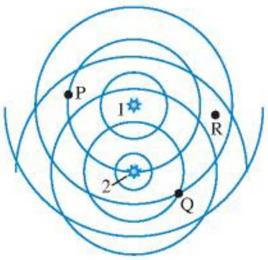


FIGURE EX17.28

28. || **FIGURE EX17.28** shows the circular wave fronts emitted by two wave sources.
    a. Are these sources in phase or out of phase? Explain.
    b. Make a table with rows labeled P, Q, and R and columns labeled r_1, r_2, Δr, and C/D. Fill in the table for points P, Q, and R, giving the distances as multiples of λ and indicating, with a C or a D, whether the interference at that point is constructive or destructive.
29. || Two in-phase loudspeakers, which emit sound in all directions, are sitting side by side. One of them is moved sideways by 3.0 m, then forward by 4.0 m. Afterward, constructive interference is observed $\frac{1}{4}$ and $\frac{3}{4}$ of the distance between the speakers along the line that joins them. What is the maximum possible wavelength of the sound waves?
30. || Two in-phase speakers 2.0 m apart in a plane are emitting 1800 Hz sound waves into a room where the speed of sound is 340 m/s. Is the point 4.0 m in front of one of the speakers, perpendicular to the plane of the speakers, a point of maximum constructive interference, maximum destructive interference, or something in between?
31. || Two out-of-phase radio antennas at $x = \pm 300$ m on the x-axis are emitting 3.0 MHz radio waves. Is the point $(x, y) = (300 \text{ m}, 800 \text{ m})$ a point of maximum constructive interference, maximum destructive interference, or something in between?

Section 17.8 Beats

32. | Two strings are adjusted to vibrate at exactly 200 Hz. Then the tension in one string is increased slightly. Afterward, three beats per second are heard when the strings vibrate at the same time. What is the new frequency of the string that was tightened?
33. | A flute player hears four beats per second when she compares her note to a 523 Hz tuning fork (the note C). She can match the frequency of the tuning fork by pulling out the "tuning joint" to lengthen her flute slightly. What was her initial frequency?
34. | Traditional Indonesian music uses an ensemble called a *gamelan* that is based on tuned percussion instruments somewhat like gongs. In Bali, the gongs are often grouped in pairs that are slightly out of tune with each other. When both are played at once, the beat frequency lends a distinctive vibrating quality to the music. Suppose a pair of gongs are tuned to produce notes at 151 Hz and 155 Hz. How many beats are heard if the gongs are struck together and both ring for 2.5 s?
35. || Two microwave signals of nearly equal wavelengths can generate a beat frequency if both are directed onto the same microwave detector. In an experiment, the beat frequency is 100 MHz. One microwave generator is set to emit microwaves with a wavelength of 1.250 cm. If the second generator emits the longer wavelength, what is that wavelength?

Problems

36. | A 2.0-m-long string vibrates at its second-harmonic frequency with a maximum amplitude of 2.0 cm. One end of the string is at $x = 0$ cm. Find the oscillation amplitude at $x = 10$, 20, 30, 40, and 50 cm.
37. || A string vibrates at its third-harmonic frequency. The amplitude at a point 30 cm from one end is half the maximum amplitude. How long is the string?
38. || **BIO** Tendons are, essentially, elastic cords stretched between two fixed ends. As such, they can support standing waves. A woman has a 20-cm-long Achilles tendon—connecting the heel to a muscle in the calf—with a cross-section area of 90 mm^2. The density of tendon tissue is 1100 kg/m^3. For a reasonable tension of 500 N, what will be the fundamental frequency of her Achilles tendon?
39. || **BIO** Biologists think that some spiders "tune" strands of their web to give enhanced response at frequencies corresponding to those at which desirable prey might struggle. Orb spider web silk has a typical diameter of 20 μm, and spider silk has a density of

1300 kg/m^3. To have a fundamental frequency at 100 Hz, to what tension must a spider adjust a 12-cm-long strand of silk?

40. || A particularly beautiful note reaching your ear from a rare Stradivarius violin has a wavelength of 39.1 cm. The room is slightly warm, so the speed of sound is 344 m/s. If the string's linear density is 0.600 g/m and the tension is 150 N, how long is the vibrating section of the violin string?

41. || A violinist places her finger so that the vibrating section of a 1.0 g/m string has a length of 30 cm, then she draws her bow across it. A listener nearby in a 20°C room hears a note with a wavelength of 40 cm. What is the tension in the string?

42. || A steel wire is used to stretch the spring of **FIGURE P17.42**. An oscillating magnetic field drives the steel wire back and forth. A standing wave with three antinodes is created when the spring is stretched 8.0 cm. What stretch of the spring produces a standing wave with two antinodes?

FIGURE P17.42

43. || Astronauts visiting Planet X have a 250-cm-long string whose mass is 5.00 g. They tie the string to a support, stretch it horizontally over a pulley 2.00 m away, and hang a 4.00 kg mass on the free end. Then the astronauts begin to excite standing waves on the horizontal portion of the string. Their data are as follows:

m	Frequency (Hz)
1	31
2	66
3	95
4	130
5	162

Use the best-fit line of an appropriate graph to determine the value of g, the free-fall acceleration on Planet X.

44. || A 75 g bungee cord has an equilibrium length of 1.20 m. The cord is stretched to a length of 1.80 m, then vibrated at 20 Hz. This produces a standing wave with two antinodes. What is the spring constant of the bungee cord?

45. || A metal wire under tension T_0 vibrates at its fundamental frequency. For what tension will the second-harmonic frequency be the same as the fundamental frequency at tension T_0?

46. ||| In a laboratory experiment, one end of a horizontal string is tied to a support while the other end passes over a frictionless pulley and is tied to a 1.5 kg sphere. Students determine the frequencies of standing waves on the horizontal segment of the string, then they raise a beaker of water until the hanging 1.5 kg sphere is completely submerged. The frequency of the fifth harmonic with the sphere submerged exactly matches the frequency of the third harmonic before the sphere was submerged. What is the diameter of the sphere?

47. ||| A vibrating standing wave on a string radiates a sound wave with intensity proportional to the square of the standing-wave amplitude. When a piano key is struck and held down, so that the string continues to vibrate, the sound level decreases by 8.0 dB in 1.0 s. What is the string's damping time constant τ?

48. || What is the fundamental frequency of the steel wire in **FIGURE P17.48**?

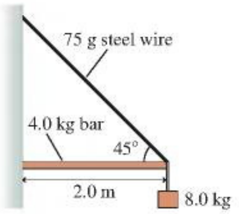


FIGURE P17.48

49. || The two strings in **FIGURE P17.49** are of equal length and are being driven at equal frequencies. The linear density of the left string is 5.0 g/m. What is the linear density of the right string?

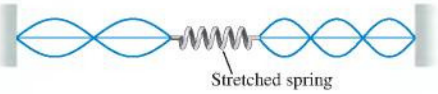


FIGURE P17.49

50. || Western music uses a musical scale with *equal temperament* tuning, which means that any two adjacent notes have the same frequency ratio r. That is, notes n and $n+1$ are related by $f_{n+1} = rf_n$ for all n. In this system, the frequency doubles every 12 notes—an interval called an *octave*.
    a. What is the value of r?
    b. Orchestras tune to the note A, which has a frequency of 440 Hz. What is the frequency of the next note of the scale (called A-sharp)?

51. || An open-open organ pipe is 78.0 cm long. An open-closed pipe has a fundamental frequency equal to the third harmonic of the open-open pipe. How long is the open-closed pipe?

52. || **BIO** Deep-sea divers often breathe a mixture of helium and oxygen to avoid getting the "bends" from breathing high-pressure nitrogen. The helium has the side effect of making the divers' voices sound odd. Although your vocal tract can be roughly described as an open-closed tube, the way you hold your mouth and position your lips greatly affects the standing-wave frequencies of the vocal tract. This is what allows different vowels to sound different. The "ee" sound is made by shaping your vocal tract to have standing-wave frequencies at, normally, 270 Hz and 2300 Hz. What will these frequencies be for a helium-oxygen mixture in which the speed of sound at body temperature is 750 m/s? The speed of sound in air at body temperature is 350 m/s.

53. || In 1866, the German scientist Adolph Kundt developed a technique for accurately measuring the speed of sound in various gases. A long glass tube, known today as a Kundt's tube, has a vibrating piston at one end and is closed at the other. Very finely ground particles of cork are sprinkled in the bottom of the tube before the piston is inserted. As the vibrating piston is slowly moved forward, there are a few positions that cause the cork particles to collect in small, regularly spaced piles along the bottom. **FIGURE P17.53** shows an experiment in which the tube is filled with pure oxygen and the piston is driven at 400 Hz. What is the speed of sound in oxygen?

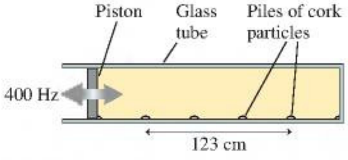


FIGURE P17.53

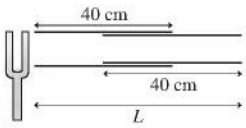


FIGURE P17.54

54. || The 40-cm-long tube of FIGURE P17.54 has a 40-cm-long insert that can be pulled in and out. A vibrating tuning fork is held next to the tube. As the insert is slowly pulled out, the sound from the tuning fork creates standing waves in the tube when the total length L is 42.5 cm, 56.7 cm, and 70.9 cm. What is the frequency of the tuning fork? Assume $v_{\text{sound}} = 343$ m/s.

55. || A 1.0-m-tall vertical tube is filled with 20°C water. A tuning fork vibrating at 580 Hz is held just over the top of the tube as the water is slowly drained from the bottom. At what water heights, measured from the bottom of the tube, will there be a standing wave in the tube above the water?

56. ||| A 44-cm-diameter water tank is filled with 35 cm of water. A
CALC 3.0-mm-diameter spigot at the very bottom of the tank is opened and water begins to flow out. The water falls into a 2.0-cm-diameter, 40-cm-tall glass cylinder. As the water falls and creates noise, resonance causes the column of air in the cylinder to produce a tone at the column's fundamental frequency. What are (a) the frequency and (b) the rate at which the frequency is changing (Hz/s) when the cylinder has been filling for 4.0 s? You can assume that the height of the water in the tank does not appreciably change in 4.0 s.

57. || A 25-cm-long wire with a linear density of 20 g/m passes across the open end of an 85-cm-long open-closed tube of air. If the wire, which is fixed at both ends, vibrates at its fundamental frequency, the sound wave it generates excites the second vibrational mode of the tube of air. What is the tension in the wire? Assume $v_{\text{sound}} = 340$ m/s.

58. || An old mining tunnel disappears into a hillside. You would like to know how long the tunnel is, but it's too dangerous to go inside. Recalling your recent physics class, you decide to try setting up standing-wave resonances inside the tunnel. Using your subsonic amplifier and loudspeaker, you find resonances at 4.5 Hz and 6.3 Hz, and at no frequencies between these. It's rather chilly inside the tunnel, so you estimate the sound speed to be 335 m/s. Based on your measurements, how far is it to the end of the tunnel?

59. || Two in-phase loudspeakers emit identical 1000 Hz sound waves along the x-axis. What distance should one speaker be placed behind the other for the sound to have an amplitude 1.5 times that of each speaker alone?

60. || Analyze the standing sound waves in an open-closed tube to show that the possible wavelengths and frequencies are given by Equation 17.18.

61. || Two loudspeakers emit sound waves of the same frequency along the x-axis. The amplitude of each wave is a. The sound intensity is minimum when speaker 2 is 10 cm behind speaker 1. The intensity increases as speaker 2 is moved forward and first reaches maximum, with amplitude $2a$, when it is 30 cm in front of speaker 1. What is
   a. The wavelength of the sound?
   b. The phase difference between the two loudspeakers?
   c. The amplitude of the sound (as a multiple of a) if the speakers are placed side by side?

62. || Two loudspeakers emit sound waves along the x-axis. A listener in front of both speakers hears a maximum sound intensity when speaker 2 is at the origin and speaker 1 is at $x = 0.50$ m. If speaker 1 is slowly moved forward, the sound intensity decreases and then increases, reaching another maximum when speaker 1 is at $x = 0.90$ m.
   a. What is the frequency of the sound? Assume $v_{\text{sound}} = 340$ m/s.
   b. What is the phase difference between the speakers?

63. || A sheet of glass is coated with a 500-nm-thick layer of oil ($n = 1.42$).
   a. For what *visible* wavelengths of light do the reflected waves interfere constructively?
   b. For what *visible* wavelengths of light do the reflected waves interfere destructively?
   c. What is the color of reflected light? What is the color of transmitted light?

64. || A manufacturing firm has hired your company, Acoustical Consulting, to help with a problem. Their employees are complaining about the annoying hum from a piece of machinery. Using a frequency meter, you quickly determine that the machine emits a rather loud sound at 1200 Hz. After investigating, you tell the owner that you cannot solve the problem entirely, but you can at least improve the situation by eliminating reflections of this sound from the walls. You propose to do this by installing mesh screens in front of the walls. A portion of the sound will reflect from the mesh; the rest will pass through the mesh and reflect from the wall. How far should the mesh be placed in front of the wall for this scheme to work?

65. || A soap bubble is essentially a very thin film of water ($n = 1.33$) surrounded by air. The colors that you see in soap bubbles are produced by interference.
   a. Derive an expression for the wavelengths λ_C for which constructive interference causes a strong reflection from a soap bubble of thickness d.

   Hint: Think about the reflection phase shifts at both boundaries.
   b. What visible wavelengths of light are strongly reflected from a 390-nm-thick soap bubble? What color would such a soap bubble appear to be?

66. || Engineers are testing a new thin-film coating whose index of refraction is less than that of glass. They deposit a 560-nm-thick layer on glass, then shine lasers on it. A red laser with a wavelength of 640 nm has no reflection at all, but a violet laser with a wavelength of 400 nm has a maximum reflection. How the coating behaves at other wavelengths is unknown. What is the coating's index of refraction?

67. || Scientists are testing a transparent material whose index of refraction for visible light varies with wavelength as $n = 30.0\ \text{nm}^{1/2}/\lambda^{1/2}$, where λ is in nm. If a 295-nm-thick coating is placed on glass ($n = 1.50$) for what visible wavelengths will the reflected light have maximum constructive interference?

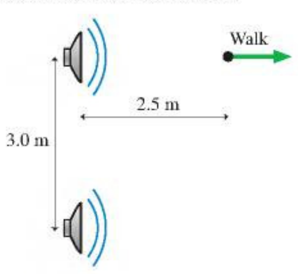


FIGURE P17.68

68. || You are standing 2.5 m directly in front of one of the two loudspeakers shown in FIGURE P17.68. They are 3.0 m apart and both are playing a 686 Hz tone in phase. As you begin to walk directly away from the speaker, at what distances from the speaker do you hear a *minimum* sound intensity? The room temperature is 20°C.

69. || Two loudspeakers in a plane, 5.0 m apart, are playing the same frequency. If you stand 12.0 m in front of the plane of the speakers, centered between them, you hear a sound of maximum intensity. As you walk parallel to the plane of the speakers, staying 12.0 m in front of them, you first hear a minimum of sound intensity when you are directly in front of one of the speakers. What is the frequency of the sound? Assume a sound speed of 340 m/s.

70. || Two identical loudspeakers separated by distance Δx each emit sound waves of wavelength λ and amplitude a along the x-axis. What is the minimum value of the ratio $\Delta x/\lambda$ for which the amplitude of their superposition is also a?

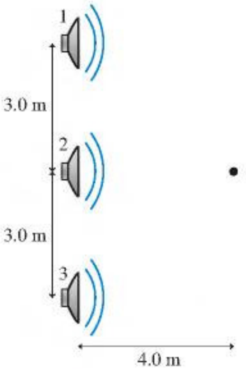


FIGURE P17.71

71. || The three identical loudspeakers in FIGURE P17.71 play a 170 Hz tone in a room where the speed of sound is 340 m/s. You are standing 4.0 m in front of the middle speaker. At this point, the amplitude of the wave from each speaker is a.
    a. What is the amplitude at this point?
    b. How far must speaker 2 be moved to the left to produce a maximum amplitude at the point where you are standing?
    c. When the amplitude is maximum, by what factor is the sound intensity greater than the sound intensity from a single speaker?
72. | Piano tuners tune pianos by listening to the beats between the *harmonics* of two different strings. When properly tuned, the note A should have a frequency of 440 Hz and the note E should be at 659 Hz.
    a. What is the frequency difference between the third harmonic of the A and the second harmonic of the E?
    b. A tuner first tunes the A string very precisely by matching it to a 440 Hz tuning fork. She then strikes the A and E strings simultaneously and listens for beats between the harmonics. What beat frequency indicates that the E string is properly tuned?
    c. The tuner starts with the tension in the E string a little low, then tightens it. What is the frequency of the E string when she hears four beats per second?
73. || A flutist assembles her flute in a room where the speed of sound is 342 m/s. When she plays the note A, it is in perfect tune with a 440 Hz tuning fork. After a few minutes, the air inside her flute has warmed to where the speed of sound is 346 m/s.
    a. How many beats per second will she hear if she now plays the note A as the tuning fork is sounded?
    b. How far does she need to extend the "tuning joint" of her flute to be in tune with the tuning fork?
74. || **CALC** You have two small, identical boxes that generate 440 Hz notes. While holding one, you drop the other from a 20-m-high balcony. How many beats will you hear before the falling box hits the ground? You can ignore air resistance.
75. || Two loudspeakers emit 400 Hz notes. One speaker sits on the ground. The other speaker is in the back of a pickup truck. You hear eight beats per second as the truck drives away from you. What is the truck's speed?

Challenge Problems

76. ||| Two radio antennas are separated by 2.0 m. Both broadcast identical 750 MHz waves. If you walk around the antennas in a circle of radius 10 m, how many maxima will you detect?
77. ||| A 280 Hz sound wave is directed into one end of the trombone slide seen in FIGURE CP17.77. A microphone is placed at the other end to record the intensity of sound waves that are transmitted through the tube. The straight sides of the slide are 80 cm in length and 10 cm apart with a semicircular bend at the end. For what slide extensions s will the microphone detect a maximum of sound intensity?

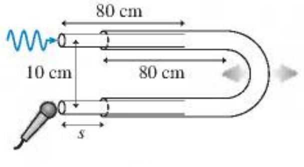


FIGURE CP17.77

78. ||| As the captain of the scientific team sent to Planet Physics, one of your tasks is to measure g. You have a long, thin wire labeled 1.00 g/m and a 1.25 kg weight. You have your accurate space cadet chronometer but, unfortunately, you seem to have forgotten a meter stick. Undeterred, you first find the midpoint of the wire by folding it in half. You then attach one end of the wire to the wall of your laboratory, stretch it horizontally to pass over a pulley at the midpoint of the wire, then tie the 1.25 kg weight to the end hanging over the pulley. By vibrating the wire, and measuring time with your chronometer, you find that the wire's second-harmonic frequency is 100 Hz. Next, with the 1.25 kg weight still tied to one end of the wire, you attach the other end to the ceiling to make a pendulum. You find that the pendulum requires 314 s to complete 100 oscillations. Pulling out your trusty calculator, you get to work. What value of g will you report back to headquarters?
79. ||| When mass M is tied to the bottom of a long, thin wire suspended from the ceiling, the wire's second-harmonic frequency is 200 Hz. Adding an additional 1.0 kg to the hanging mass increases the second-harmonic frequency to 245 Hz. What is M?
80. ||| **BIO** Ultrasound has many medical applications, one of which is to monitor fetal heartbeats by reflecting ultrasound off a fetus in the womb.
    a. Consider an object moving at speed v_o toward an at-rest source that is emitting sound waves of frequency f_0. Show that the reflected wave (i.e., the echo) that returns to the source has a Doppler-shifted frequency

    $$f_{\text{echo}} = \left(\frac{v + v_o}{v - v_o}\right) f_0$$

    where v is the speed of sound in the medium.
    b. Suppose the object's speed is much less than the wave speed: $v_o \ll v$. Then $f_{\text{echo}} \approx f_0$, and a microphone that is sensitive to these frequencies will detect a beat frequency if it listens to f_0 and f_{echo} simultaneously. Use the binomial approximation and other appropriate approximations to show that the beat frequency is $f_{\text{beat}} \approx (2v_o/v)f_0$.
    c. The reflection of 2.40 MHz ultrasound waves from the surface of a fetus's beating heart is combined with the 2.40 MHz wave to produce a beat frequency that reaches a maximum of 65 Hz. What is the maximum speed of the surface of the heart? The speed of ultrasound waves within the body is 1540 m/s.
    d. Suppose the surface of the heart moves in simple harmonic motion at 90 beats/min. What is the amplitude in mm of the heartbeat?
81. ||| A water wave is called a *deep-water wave* if the water's depth is more than one-quarter of the wavelength. Unlike the waves we've considered in this chapter, the speed of a deep-water wave depends on its wavelength:

    $$v = \sqrt{\frac{g\lambda}{2\pi}}$$

    Longer wavelengths travel faster. Let's apply this to standing waves. Consider a diving pool that is 5.0 m deep and 10.0 m wide. Standing water waves can set up across the width of the pool. Because water sloshes up and down at the sides of the pool, the boundary conditions require antinodes at $x = 0$ and $x = L$. Thus a standing water wave resembles a standing sound wave in an open-open tube.
    a. What are the wavelengths of the first three standing-wave modes for water in the pool? Do they satisfy the condition for being deep-water waves?
    b. What are the wave speeds for each of these waves?
    c. Derive a general expression for the frequencies f_m of the possible standing waves. Your expression should be in terms of m, g, and L.
    d. What are the oscillation *periods* of the first three standing wave modes?

PART IV

KNOWLEDGE STRUCTURE

Oscillations and Waves

KEY FINDINGS What are the overarching findings of Part IV?

- Particles are
  - Localized
  - Discrete
  - Two particles cannot occupy the same point in space.
- Waves are
  - Diffuse
  - Spread out
  - Two waves can pass through each other.

LAWS What laws of physics govern oscillations and waves?

Newton's second law	SHM: $d^2x/dt^2 = -\omega^2 x$	Wave equation: $\partial^2 D/\partial t^2 = v^2 \partial^2 D/\partial x^2$
Conservation of energy	For SHM: $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}m(v_{\max})^2 = \frac{1}{2}kA^2$	
Fundamental relationship for sinusoidal waves	$v = \lambda f = \omega/k$	
Principle of superposition	The net displacement is the sum of the displacements due to each wave.	

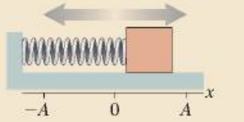
MODELS What are the most important models of Part IV?

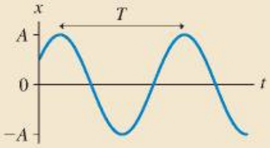
Simple harmonic motion

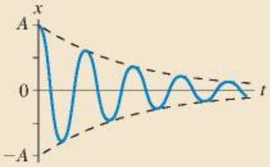
- Any object with a **linear restoring force** can undergo SHM. This is sinusoidal motion with

$$x = A\cos(\omega t + \phi_0)$$
$$v = -v_{\max}\sin(\omega t + \phi_0)$$

- Mechanical energy is conserved if there's no friction.
- With friction, the oscillations are damped. A simple model of **damping** predicts oscillations that decay exponentially with time.
- **Resonance** is a large-amplitude response when an oscillator is driven at its natural frequency.





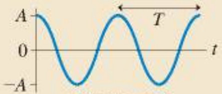


Waves

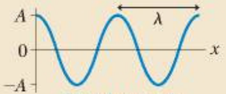
- A wave is a disturbance that travels.
  - **Mechanical waves** travel through a medium.
  - **Electromagnetic waves** travel through a vacuum.
  - Waves can be **transverse** or **longitudinal.**
- Wave speed is a property of the medium.
- Sinusoidal waves are periodic in both time (period) and space (wavelength). They obey $v = \lambda f$.
- Waves obey the principle of superposition.

Transverse wave

Longitudinal wave



History graph



Snapshot graph

TOOLS What are the most important tools introduced in Part IV?

Oscillation period

- Frequency $f = 1/T$
- Angular frequency $\omega = 2\pi f = 2\pi/T$
- Spring $T = 2\pi\sqrt{m/k}$
- Pendulum $T = 2\pi\sqrt{L/g}$
- Wave $f = v/\lambda$

Sinusoidal wave

- Displacement is a function of x and t:

$$D(x,t) = A\sin(kx \mp \omega t + \phi_0)$$

- $-\omega t$ for motion to the right
- $+\omega t$ for motion to the left
- The *wave number* is $k = 2\pi/\lambda$

Sound intensity level

$$\beta = (10\text{ dB})\log_{10}(I/10^{-12}\text{ W/m}^2)$$

Wave speed

- String $v = \sqrt{T_s/\mu}$
- Sound $v = \sqrt{B/\rho}$

Phase

- The quantity $\omega t + \phi_0$ is called the phase ϕ of SHM.
- The quantity $k - \omega t + \phi_0$ is the phase ϕ of a sinusoidal wave.
- The *phase constant* ϕ_0 is given by the initial conditions.

Doppler effect

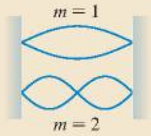
A frequency shift when the source moves relative to an observer:

$$f = f_0/(1 \mp v_s/v)$$

for an approaching/receding source.

Standing waves

Two waves traveling in opposite directions.



- Strings, open-open tubes, and closed-closed tubes:

$$f = mf_1$$
$$f_1 = v/2L = \text{fundamental frequency}$$

- Standing waves have points that never move called *nodes*.

Interference

- *Constructive interference* if *in phase*

$$\Delta\phi = m \cdot 2\pi \text{ rad}$$

- *Destructive interference* if *out of phase*

$$\Delta\phi = \left(m + \tfrac{1}{2}\right) \cdot 2\pi \text{ rad}$$

- *Beats* $f_{\text{beat}} = |f_1 - f_2|$ if frequencies differ.

PART

V Thermodynamics

Smoke particles allow us to visualize *convection,* one of the ways in which heat is transferred from one place to another.

OVERVIEW

It's All About Energy

Thermodynamics—the science of energy in its broadest context—arose hand in hand with the industrial revolution as the systematic study of converting heat energy into mechanical motion and work. Hence the name *thermo* + *dynamics*. Indeed, the analysis of engines and generators of various kinds remains the focus of engineering thermodynamics. But thermodynamics, as a science, now extends to all forms of energy conversions, including those involving living organisms. For example:

- **Engines** convert the energy of a fuel into the mechanical energy of moving pistons, gears, and wheels.
- **Fuel cells** convert chemical energy into electrical energy.
- **Photovoltaic cells** convert the electromagnetic energy of light into electrical energy.
- **Lasers** convert electrical energy into the electromagnetic energy of light.
- **Organisms** convert the chemical energy of food into a variety of other forms of energy, including kinetic energy, sound energy, and thermal energy.

The major goals of Part V are to understand both *how* energy transformations such as these take place and *how efficient* they are. We'll discover that the laws of thermodynamics place limits on the efficiency of energy transformations, and understanding these limits is essential for analyzing the very real energy needs of society in the 21st century.

Our ultimate destination in Part V is an understanding of the thermodynamics of *heat engines*. A heat engine is a device, such as a power plant or an internal combustion engine, that transforms heat energy into useful work. These are the devices that power our modern society.

Understanding how to transform heat into work will be a significant achievement, but we first have many steps to take along the way. We need to understand the concepts of temperature and pressure. We need to learn about the properties of solids, liquids, and gases. Most important, we need to expand our view of energy to include *heat,* the energy that is transferred between two systems at different temperatures.

At a deeper level, we need to see how these concepts are connected to the underlying microphysics of randomly moving molecules. We will find that the familiar concepts of thermodynamics, such as temperature and pressure, have their roots in atomic-level motion and collisions. This *micro/macro connection* will lead to the second law of thermodynamics, one of the most subtle but also one of the most profound and far-reaching statements in physics.

Only after all these steps have been taken will we be able to analyze a real heat engine. It is an ambitious goal, but one we can achieve.